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Formal integration of the equations of quantum theory in the Heisenberg representation

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With 13 figures in the text

Summary

The differential equations of quantum electrodynamics and meson theory are formally integrated in the higher approximations without use of the interaction representation. It is first shown, that the commutator formulae of Schwinger and the P-symbols of Dyson satisfy the recurrence formulae in the Heisenberg representation, and the explicit calculations needed are at least not more complicated than before. This method of integration, however, has the disadvantage of not avoiding the complications involved in the normals of the space-like surfaces, as we have to make an explicit use of the interaction Hamiltonian. The real advantage of the Heisenberg representation is thus lost. In the second part the differential equations are therefore solved directly by being substituted into themselves. Explicit formulae are given in the simplest cases and the analysis is carried so far, that the general structure also of the more complicated expressions can be inferred. The calculations are at this point a little more delicate than the use of the $S_F(x)$ and $D_F(x)$ functions (at least for particles of spin $\frac{1}{2}$, where no surface terms appear) but give, on the other hand, an interesting insight into some of the prescriptions of the earlier method. At last the theory for particles of integer spin is also investigated, and the similarity to the case of spin $\frac{1}{2}$ is stressed. In the appendix the removal of a special form of indeterminacy in the theory is illustrated by a simple example.

1. Introduction

In a previous paper[1] the author has developed a method of solving the differential equations of quantum electrodynamics (including the renormalization of charge and mass) without using the interaction representation. In (I) a few simple examples were discussed, and it was shown that the formulae obtained agreed with those derived by Schwinger and others. In this paper we want to examine the formal solutions a little more closely in the higher approximations and to compare the results of the investigation with the calculations of

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[1] G. Källén, Arkiv för fysik Bd. *2*, Nr. 19 (1950), here quoted as (I).

DYSON.[2, 3] It will appear, that our solution is always real, and that it agrees with the formulae of DYSON, when no real processes can take place.

The method referred to above consists essentially in expanding every operator (in the HEISENBERG representation) in a power series in the coupling constant and to substitute the expressions into the differential equations. If the coefficients of every power of e is then put equal to zero, we obtain ((I) equations (25) and (26))

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi^{(n+1)}(x) = \frac{i}{2} \sum_{m=0}^{n} \left\{A_\nu^{(m)}(x), \gamma_\nu \psi^{(n-m)}(x)\right\} + \sum_{m=2}^{n+1} \delta m^{(m)} \psi^{(n+1-m)}(x) \quad (1)$$

$$\Box A_\mu^{(n+1)}(x) = -\frac{i}{2} \sum_{m=0}^{n} [\bar{\psi}^{(m)}(x), \gamma_\mu \psi^{(n-m)}(x)]. \quad (2)$$

These equations are the starting point for the discussion given below, and we want to study their formal solution in some details.

The notations used are essentially the same as those of SCHWINGER and DYSON. We thus denote the space-coordinates by (x_1, x_2, x_3) and use $x_4 = i x_0 = i t$ as time coordinate. The four-vectors will not be distinguished in any special way, as in any case they are generally easily recognized. The scalar product between two such vectors a and b is simply written as

$$a b = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_0 b_0.$$

However, if we want especially to emphasize the spatial part of some four-vector, we denote it by a bar

$$\bar{a} = (a_1, a_2, a_3) \text{ and } a b = \bar{a} \bar{b} - a_0 b_0.$$

The four-dimensional integration element is, as long as no error may arise, simply denoted by $d x$. An expression of the form

$$\int d p \frac{e^{i p x}}{p^2 + m^2}$$

is thus to be interpreted as

$$\int\!\!\int\!\!\int\!\!\int d p_1 \, d p_2 \, d p_3 \, d p_0 \frac{e^{i (p_1 x_1 + p_2 x_2 + p_3 x_3 - p_0 x_0)}}{p_1^2 + p_2^2 + p_3^2 - p_0^2 + m^2}$$

and so on.

As the diagonal elements of the metric tensor are all equal to one (and the non-diagonal equal to zero) we do not distinguish between covariant and contravariant indices.

The matrices γ_ν are the well-known ones, with four rows and four columns, fulfilling

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[2] F. J. DYSON, Phys. Rev. *75*, 486 (1949).

[3] F. J. DYSON, Phys. Rev. *75*, 1736 (1949).

$$\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = 2\, g_{\mu\nu} = 2\, \delta_{\mu\nu}$$

and *e.g.* the expression

$$\gamma \frac{\partial}{\partial x} \psi(x)$$

has to be read as

$$\sum_{\nu,\,\beta} (\gamma_\nu)_{\alpha\beta} \frac{\partial \psi_\beta(x)}{\partial x_\nu}.$$

etc. The $A_\mu(x)$ operators are the electromagnetic potentials, the $\psi(x)$ operator the usual DIRAC spinor and $\psi^*(x)$ its Hermitian conjugate. The quantity $\bar{\psi}(x)$ is defined as

$$\bar{\psi}(x) = \psi^*(x)\, \gamma_4.$$

The commutator

$$[\bar{\psi}(x),\, \gamma_\nu\, \psi(x)]$$

stands for

$$\bar{\psi}(x)\, \gamma_\nu\, \psi(x) - \psi(x)\, \gamma_\nu^T\, \bar{\psi}(x) = \sum_{\alpha,\,\beta} (\gamma_\nu)_{\alpha\beta} \left(\bar{\psi}_\alpha(x)\, \psi_\beta(x) - \psi_\beta(x)\, \bar{\psi}_\alpha(x)\right).$$

The other notations are either obvious or explained in the text. Natural units ($\hbar = c = 1$) are used throughout.

2. The singular functions

In what follows we will frequently make use of the various singular functions defined by SCHWINGER and others. For the convenience of the reader a short exposé of these functions will be given here, although we have nothing essentially new to say at this point.

We begin with the function $\overline{\triangle}(x)$ defined, more or less arbitrarily, as the solution of

$$(\Box - m^2)\, \overline{\triangle}(x) = -\delta(x) = -\frac{1}{(2\pi)^4} \int e^{ipx}\, dp \tag{3}$$

which is given by

$$\overline{\triangle}(x) = \frac{1}{(2\pi)^4}\, P \int \frac{e^{ipx}\, dp}{p^2 + m^2} \tag{4}$$

(compare SCHWINGER[4], appendix). The letter P in front of the integral sign indicates, that the principal value of the integral has to be taken in the points, where the denominator vanishes. The p-integrations have to be performed in the following order. The p_0-integration is done at first, and the principal value is taken at the poles. After that, the integrations over the three-dimensional angles are carried through and, finally, we may then integrate over the absolute value of the three-dimensional vector $\bar{p}$.

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[4] J. SCHWINGER, Phys. Rev. *75*, 651 (1949).

It is now convenient to write

$$\overline{\triangle}(x) = \tfrac{1}{2}\left[\triangle_R(x) + \triangle_A(x)\right] \tag{5}$$

where the two new functions $\triangle_R(x)$ and $\triangle_A(x)$ are given by

$$\triangle_R(x) = \frac{1}{(2\pi)^4} \int\limits_{C_R} \frac{e^{ipx}}{p^2 + m^2}\, dp \tag{6}$$

$$\triangle_A(x) = \frac{1}{(2\pi)^4} \int\limits_{C_A} \frac{e^{ipx}}{p^2 + m^2}\, dp\,. \tag{7}$$

The p_0-integrations are here performed along the two complex contours shown in figure 1.

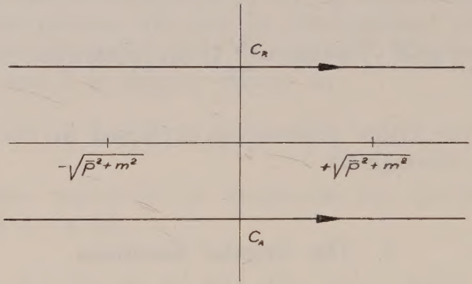

Fig. 1.

We will also make use of the functions

$$\triangle(x) = \triangle_A(x) - \triangle_R(x) = \frac{1}{(2\pi)^4} \int\limits_{C} \frac{e^{ipx}}{p^2 + m^2}\, dp \tag{8}$$

and

$$\triangle^{(1)}(x) = \frac{1}{i}\frac{1}{(2\pi)^4} \int\limits_{C_1} \frac{e^{ipx}}{p^2 + m^2}\, dp \tag{9}$$

where the paths C and C_1 are illustrated in figures 2 and 3.

The factor $\frac{1}{i}$ in (9) is inserted to secure agreement with the notations of SCHWINGER. Obviously, all these functions are not independent, and we have

$$\triangle_R(x) = \overline{\triangle}(x) - \tfrac{1}{2}\triangle(x) \tag{10}$$

$$\triangle_A(x) = \overline{\triangle}(x) + \tfrac{1}{2}\triangle(x)\,. \tag{11}$$

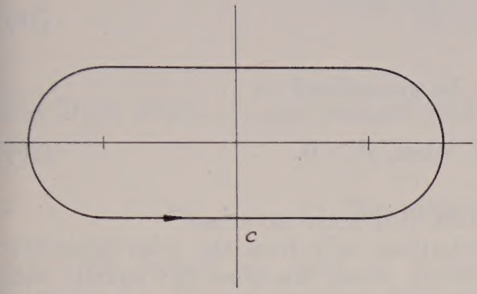

Fig. 2.

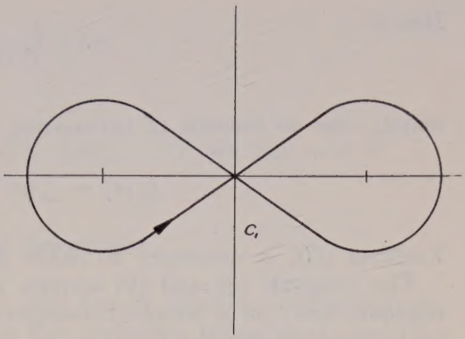

Fig. 3.

Further

$$(\Box - m^2)\,\triangle_R(x) = (\Box - m^2)\,\triangle_A(x) = -\,\delta(x) \tag{12}$$

$$(\Box - m^2)\,\triangle(x) = (\Box - m^2)\,\triangle^{(1)}(x) = 0. \tag{13}$$

Due to the factor $e^{ikx} = e^{i\bar{k}\bar{x} - ik_0 x_0}$ the integral over a circle of very large radius will disappear in the upper half-plane if $x_0 < 0$ and in the lower one if $x_0 > 0$. This means that

$$\triangle_A(x) = 0 \tag{14 a}$$

$$\overline{\triangle}(x) = -\,\tfrac{1}{2}\,\triangle(x) \tag{14 b}$$

$$\triangle_R(x) = -\,\triangle(x) \tag{14 c}$$

if $x_0 > 0$ and

$$\triangle_R(x) = 0 \tag{15 a}$$

$$\overline{\triangle}(x) = \tfrac{1}{2}\,\triangle(x) \tag{15 b}$$

$$\triangle_A(x) = \triangle(x) \tag{15 c}$$

if $x_0 < 0$. Hence

$$\overline{\triangle}(x) = -\,\tfrac{1}{2}\,\varepsilon(x)\,\triangle(x) \tag{16}$$

where

$$\varepsilon(x) = \frac{x_0}{|x_0|}. \tag{16 a}$$

From the integral representations (4) and (8) we further infer, not only that

$$\triangle(-x) = -\,\triangle(x) \tag{17}$$

$$\overline{\triangle}(-x) = \overline{\triangle}(x) \tag{17 a}$$

but also that

$$\triangle(\bar{x}, -x_0) = -\,\triangle(\bar{x}, x_0). \tag{18}$$

Hence

$$\triangle(\bar{x}, 0) = 0 \tag{19}$$

which, due to reasons of invariance, can be generalized to

$$\triangle(x) = \overline{\triangle}(x) = 0, \text{ when } x^2 > 0. \tag{20}$$

Formula (20) is necessary to make equation (16) truly invariant.[4]

The integrals (8) and (9) acquire contributions only from the poles inside the contours and, as a simple investigation shows, these functions can equally well be represented by the formulae of SCHWINGER.[4]

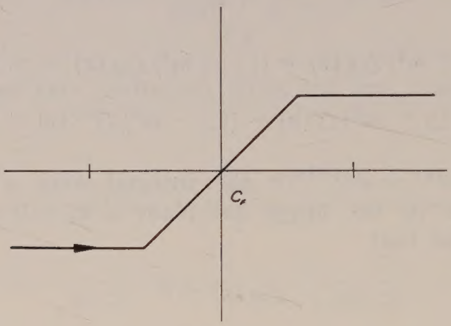

Fig. 4.

$$\triangle(x) = \frac{-i}{(2\pi)^3}\int dp\, e^{ipx}\,\delta(p^2+m^2)\,\varepsilon(p) \tag{21}$$

$$\triangle^{(1)}(x) = \frac{1}{(2\pi)^3}\int dp\, e^{ipx}\,\delta(p^2+m^2)\cdot \tag{22}$$

As last we also introduce the function

$$\triangle_F(x) = \frac{2}{i}\frac{1}{(2\pi)^4}\int\limits_{C_F}\frac{e^{ipx}}{p^2+m^2}\,dp \tag{23}$$

which has been extensively used by FEYNMAN[5] and DYSON.[2, 3]

The functions $S(x)$, $\bar{S}(x)$, . . . etc. are given by

$$S(x) = \left(\gamma\frac{\partial}{\partial x} - m\right)\triangle(x)$$

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[5] R. F. FEYNMAN, Phys. Rev. *76*, 749 (1949), ibid. *76*, 769 (1949).

$$\bar{S}(x) = \left(\gamma \frac{\partial}{\partial x} - m\right) \overline{\triangle}(x)$$

$$\cdots\cdots\cdots\cdots$$

and $D(x)$, $\bar{D}(x)$, . . . are obtained from $\triangle(x)$, $\overline{\triangle}(x)$, . . . if we put $m = 0$.

3. The solution of Schwinger

In this section we want to verify that the formal solution of SCHWINGER (compare *e.g.* DYSON[2] equation (25)) satisfies our recursion formulae (1) and (2). Although this is more or less evident, due to the fact that the two expressions have been derived from the same starting-point, it may be of some interest to consider the detailed calculations necessary. For this purpose we require a theorem, which can be stated as follows.

If $A(x)$, $B(x)$ and $H(x)$ are three operators, and if we define two sets of new operators according to the scheme

$$A^{(n)}(x) = i^n \int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-1}} dx^n \, [H(x^n), [\dots [H(x'), A(x)] \dots]] \tag{24}$$

and

$$B^{(n)}(x) = i^n \int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-1}} dx^n \, [H(x^n), [\dots [H(x'), B(x)] \dots]] \tag{25}$$

we have identically

$$(A(x)\, B(x))^{(n)} = i^n \int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-1}} dx^n \, [H(x^n), [\dots [H(x'), A(x)\, B(x)] \dots]] = \sum_{m=0}^{n} A^{(m)}(x)\, B^{(n-m)}(x). \tag{26}$$

To prove this formula we expand the iterated commutator in the following way

$$\begin{aligned}
[H(x^n), [\dots [H(x'), A(x)\, B(x)] \dots]] &= [H(x^n), [\dots [H(x'), A(x)] \dots \\
&\dots]]\, B(x) + \sum_{i=1}^{n} [H(x^n), [\dots [H(x^{i+1}), [H(x^{i-1}), [\dots [H(x'), A(x)] \dots \\
&\dots]]] \dots]]\, [H(x^i), B(x)] + \sum_{i>j} [H(x^n), [\dots [H(x^{i+1}), [H(x^{i-1}), [\dots \\
&\dots [H(x^{j+1}), [H(x^{j-1}), [\dots [H(x'), A(x)] \dots]]] \dots]]] \dots]] \cdot \\
&\cdot [H(x^i), [H(x^j), B(x)]] + \sum_{i>j>k} \dots + \dots
\end{aligned} \tag{27}$$

The first term in (27) gives, when integrated, directly $A^{(n)}(x) \cdot B(x)$. The second term gives a contribution of the following form

$$\int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-1}} dx^n \sum_{i=1}^{n} f(x^n \dots x^{i+1} x^{i-1} \dots x')\, g(x^i). \tag{28}$$

This expression, however, is equal to

$$\int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-2}} dx^{n-1} f(x^{n-1} \dots x') \int_{-\infty}^{x_0} dx^n\, g(x^n) \tag{29}$$

which is easily shown from n to $n+1$. Supposing (28) and (29) to be equal up to some value n, we get

$$\begin{aligned}
&\int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n}} dx^{n+1} \sum_{i=1}^{n+1} f(x^{n+1} \dots x^{i+1} x^{i-1} \dots x')\, g(x^i) = \\
&\quad = \int_{-\infty}^{x_0} dx^{n+1} \int_{-\infty}^{x_0^{n+1}} dx' \dots \int_{-\infty}^{x_0^{n-1}} dx^n f(x^n \dots x')\, g(x^{n+1}) + \\
&\quad + \int_{-\infty}^{x_0} dx' \int_{-\infty}^{x'_0} dx'' \dots \int_{-\infty}^{x_0^{n-1}} dx^n f(x^n \dots x') \int_{-\infty}^{x'_0} dx^{n+1} g(x^{n+1}) = \\
&\quad = \int_{-\infty}^{x_0} dx' \dots \int_{-\infty}^{x_0^{n-1}} dx^n f(x^n \dots x') \int_{-\infty}^{x_0} dx^{n+1} g(x^{n+1}).
\end{aligned} \tag{29 a}$$

The first term in (29 a) is obtained from $i = 1$ only through a renumeration of the variables. The second will appear, if we apply our theorem (which was supposed valid up to the integer n) for the variables $x'', \dots x^{n+1}$. As (28) and (29) are equal *e.g.* for $n = 2$, they are thus equal for all integers n.

In a quite analogous way we obtain, more generally

$$\begin{aligned}
&\int_{-\infty}^{x_0} dx' \dots \int_{-\infty}^{x_0^{n-1}} dx^n \sum_{i_1 > i_2 > \cdots > i_r} f(x^n \dots x^{i_1+1} x^{i_1-1} \dots \dots x^{i_r+1} x^{i_r-1} \dots x') \cdot \\
&\quad \cdot g(x^{i_1} \dots x^{i_r}) = \int_{-\infty}^{x_0} dx' \dots \int_{-\infty}^{x_0^{n-r-1}} dx^{n-r} f(x^{n-r} \dots x') \int_{-\infty}^{x_0} dx^{n-r+1} \dots \\
&\quad \dots \int_{-\infty}^{x_0^{n-1}} dx^n\, g(x^n \dots x^{n-r+1}).
\end{aligned} \tag{30}$$

Equation (30) is proved by observing that if it is valid for some integer r, we have

$$\begin{aligned}
&\int_{-\infty}^{x_0} d\,x' \ldots \int_{-\infty}^{x_0^{n-1}} d\,x^n \sum_{i_1 > \cdots > i_{r+1}} f\,(x^n \ldots x^{i_1+1}\,x^{i_1-1} \ldots \ldots x^{i_{r+1}+1}\,x^{i_{r+1}-1} \ldots x')\,\cdot \\
&\quad \cdot\, g\,(x^{i_1} \ldots x^{i_{r+1}}) = \sum_{i_{r+1}=1}^{n-r} \int_{-\infty}^{x_0} d\,x' \ldots \int_{-\infty}^{x_0^{i_{r+1}-1}} d\,x^{i_{r+1}} \int_{-\infty}^{x_0^{i_{r+1}}} d\,x^{i_{r+1}+1} \ldots \\
&\quad \ldots \int_{-\infty}^{x_0^{n-r-1}} d\,x^{n-r}\, f\,(x^{n-r} \ldots x^{i_{r+1}+1}\,x^{i_{r+1}-1} \ldots x') \int_{-\infty}^{x_0^{i_{r+1}}} d\,x^{n-r+1} \ldots \\
&\quad \ldots \int_{-\infty}^{x_0^{n-1}} d\,x^n\, g\,(x^n \ldots x^{n-r+1}\,x^{i_{r+1}}).
\end{aligned} \tag{31}$$

The right-hand side of equation (31), however, is of the same form as (28), and we infer with the aid of (29) that (30) is valid also for $r + 1$. For $r = 1$ formula (30) is proved earlier, and it is thus correct for all integers r. If we now integrate both sides of equation (27) over the $4\,n$-dimensional "tetrahedron"

$$\int_{-\infty}^{x_0} d\,x' \int_{-\infty}^{x'_0} d\,x'' \ldots \int_{-\infty}^{x_0^{n-1}} d\,x^n$$

we obtain from each term of the right-hand side an integral of the form (30), that is, a product of two operators $A^{(m)}(x)$ and $B^{(n-m)}(x)$. Obviously, all values of m less than n will appear, and we get

$$(A\,(x)\,B\,(x))^{(n)} = \sum_{m=0}^{n} A^{(m)}\,(x)\,B^{(n-m)}\,(x)$$

as desired.

After this preliminary calculation we have no difficulty in showing, that the expressions

$$\psi^{(n)}\,(x) = i^n \int_{-\infty}^{x_0} d\,x' \ldots \int_{-\infty}^{x_0^{n-1}} d\,x^n\,[H\,(x^n), [\ldots [H\,(x')\,\psi^{(0)}\,(x)] \ldots]] \tag{32}$$

and

$$A_\mu^{(n)}\,(x) = i^n \int_{-\infty}^{x_0} d\,x' \ldots \int_{-\infty}^{x_0^{n-1}} d\,x^n\,[H\,(x^n), [\ldots [H\,(x'), A_\mu^{(0)}\,(x)] \ldots]] \tag{33}$$

satisfy the recursion formulae (1) and (2), if the part containing the mass-renormalization term is dropped, and if $H\,(x)$ is given by

$$H\,(x) = -\,j_\nu^{(0)}\,(x)\,A_\nu^{(0)}\,(x). \tag{34}$$

We first consider the equation for $\psi(x)$ and obtain from

$$\int_{-\infty}^{x_0} [H(x'), \psi^{(0)}(x)]\, dx' = -\int_{-\infty}^{x_0} dx'\, A_\nu^{(0)}(x')\, \frac{i}{2} [[\bar{\psi}^{(0)}(x'), \gamma_\nu \psi^{(0)}(x')], \psi^{(0)}(x)] =$$

$$= \int_{-\infty}^{x_0} dx'\, S(x-x')\, \gamma A^{(0)}(x')\, \psi^{(0)}(x')\, dx' =$$

$$= -\int_{-\infty}^{+\infty} S_R(x-x')\, \gamma A^{(0)}(x')\, \psi^{(0)}(x')\, dx' \tag{35}$$

immediately

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi^{(n+1)}(x) = i^{n+1} \int dx'\, \delta(x-x') \int_{-\infty}^{x'_0} dx'' \ldots \int_{-\infty}^{x_0^n} dx^{n+1} [H(x^{n+1}),$$

$$[\ldots [H(x''), A_\nu^{(0)}(x')\, \gamma_\nu \psi^{(0)}(x')] \ldots]] = \frac{i}{2} \sum_{m=0}^{n} \left\{ A_\nu^{(m)}(x), \gamma_\nu \psi^{(n-m)}(x) \right\}. \tag{36}$$

The calculations for $A_\mu(x)$ are similar. We have

$$\int_{-\infty}^{x_0} dx' [H(x'), A_\mu^{(0)}(x)] = -\frac{i}{2} \int_{-\infty}^{x_0} [\bar{\psi}^{(0)}(x'), \gamma_\nu \psi^{(0)}(x')] \cdot$$

$$\cdot [A_\nu^{(0)}(x'), A_\mu^{(0)}(x)]\, dx' = -\frac{1}{2} \int_{-\infty}^{x_0} D(x-x')\, [\bar{\psi}^{(0)}(x'), \gamma_\mu \psi^{(0)}(x')]\, dx' =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} D_R(x-x')\, [\bar{\psi}^{(0)}(x'), \gamma_\mu \psi^{(0)}(x')]\, dx' \tag{37}$$

and hence

$$\Box A_\mu^{(n+1)}(x) = -\frac{i^{n+1}}{2} \int_{-\infty}^{+\infty} \delta(x-x')\, dx' \int_{-\infty}^{x'_0} dx'' \ldots$$

$$\ldots \int_{-\infty}^{x_0^n} dx^{n+1} [H(x^{n+1}), [\ldots [H(x''), [\bar{\psi}^{(0)}(x'), \gamma_\mu \psi^{(0)}(x')]] \ldots]] =$$

$$= -\frac{i}{2} \sum_{m=0}^{n} [\bar{\psi}^{(m)}(x), \gamma_\mu \psi^{(n-m)}(x)]. \tag{38}$$

The equations (32), (33) and (26) together supply the complete formalism of SCHWINGER.

4. The S-matrix of Dyson

By starting from our equations (1) and (2), we want in this paragraph to obtain the solution given by DYSON.[2] For this purpose it is most convenient to calculate the elements of the $\boldsymbol{S}$-matrix directly. This quantity satisfies the following equations, given by YANG.[6]

$$[\boldsymbol{S}, \psi^{(0)}(x)] = -ie\boldsymbol{S}\int S(x-x')\gamma A(x')\psi(x')\,dx' \tag{39}$$

$$[\boldsymbol{S}, A_\mu^{(0)}(x)] = \frac{ie}{2}\boldsymbol{S}\int D(x-x')[\bar{\psi}(x'), \gamma_\mu\psi(x')]\,dx'. \tag{40}$$

For the derivation of these formulae the reader must be referred to a forthcoming paper by YANG and FELDMAN. Expanding all operators in a power series in e, we have

$$[\boldsymbol{S}^{(n+1)}, \psi^{(0)}(x)] = -i\sum_{m=0}^{n}\boldsymbol{S}^{(m)}\int S(x-x')\gamma_\nu(A_\nu(x')\psi(x'))^{(n-m)}\,dx' \tag{41}$$

$$[\boldsymbol{S}^{(n+1)}, A_\mu^{(0)}(x)] = \frac{i}{2}\sum_{m=0}^{n}\boldsymbol{S}^{(m)}\int D(x-x')[\bar{\psi}(x'), \gamma_\mu\psi(x')]^{(n-m)}\,dx'. \tag{42}$$

DYSON'S solution is

$$\boldsymbol{S}^{(n)} = \frac{(-i)^n}{n!}\int\cdots\int dx'\ldots dx^n\,P(H(x^n)\ldots H(x')) \tag{43}$$

where $H(x)$ is given by (34) and the P-symbol is defined in the following way[2]

$$P(A(x)B(x')) = \begin{cases} A(x)B(x') & \text{when } x_0' > x_0 \\ B(x')A(x) & \text{when } x_0 > x_0'. \end{cases} \tag{44}$$

The generalization for more than two factors is obvious. From the definition of the P-symbol we find, without difficulty, for arbitrary operators $H(x)$ and $F(x)$

$$\int\cdots\int dx\,dx'\ldots dx^n\,P(H(x^n)\ldots H(x')F(x)) = \int\cdots\int dx\,dx'\ldots dx^n\cdot$$

$$\cdot P(H(x^n)\ldots H(x'))F(x) - n\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x_0}dx'\Big\{\int dx''\ldots\int dx^n\,P(H(x^n\ldots$$

$$\ldots H(x''))\cdot[H(x'), F(x)] - (n-1)\int_{-\infty}^{x'_0}dx''\Big[\int dx'''\ldots\int dx^n\,P(H(x^n)\ldots$$

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[6] Compare (I) note 3 and a forthcoming paper by YANG and FELDMAN.

$$\ldots H(x''')) [H(x''), [H(x'), F(x)]] - \cdots]\} =$$

$$= \int \cdots \int dx \ldots dx^n \, P(H(x^n) \ldots H(x')) \, F(x) -$$

$$- n \int dx \int\limits_{-\infty}^{x_0} dx' \int dx'' \ldots \int dx^n \cdot P(H(x^n) \ldots H(x'')) [H(x'), F(x)] +$$

$$+ n(n-1) \int dx \int\limits_{-\infty}^{x_0} dx' \int\limits_{-\infty}^{x'_0} dx'' \int dx''' \ldots \int dx^n \cdot$$

$$\cdot P(H(x^n) \ldots H(x''')) \cdot [H(x''), [H(x'), F(x)]] - \cdots \tag{45}$$

If we substitute equation (43) into the left-hand side of (41) and apply (45) with $F(x) = \gamma A^{(0)}(x) \, \psi^{(0)}(x)$ we obtain, using the results of the previous section

$$\frac{(-i)^{n+1}}{(n+1)!} \int \cdots \int dx' \ldots dx^{n+1} (n+1) \, P(H(x^{n+1}) \ldots H(x'') [H(x'), \psi^{(0)}(x)]) =$$

$$= \frac{(-i)^{n+1}}{n!} \int \cdots \int dx' \ldots dx^{n+1} \, S(x-x') \, P(H(x^{n+1}) \ldots H(x'') \gamma A^{(0)}(x') \, \psi^{(0)}(x')) =$$

$$= -i \sum_{m=0}^{n} \mathbf{S}^{(m)} \int dx' \, S(x-x') \, \gamma_\nu (A_\nu(x') \, \psi(x'))^{(n-m)}. \tag{46}$$

Substituting (43) into (42), we obtain in the same way

$$\frac{(-i)^{n+1}}{(n+1)!} \int \cdots \int dx' \ldots dx^{n+1} \, P(H(x^{n+1}) \ldots H(x'') [H(x'), A_\mu^{(0)}(x)]) (n+1) =$$

$$= \frac{(-i)^{n+1}}{n!} (-\tfrac{1}{2}) \int \cdots \int dx' \ldots dx^{n+1} \, D(x-x') \, P(H(x^{n+1}) \ldots H(x'') [\bar{\psi}^{(0)}(x'),$$

$$\gamma_\mu \psi^{(0)}(x')]) = \frac{i}{2} \sum_{m=0}^{n} \mathbf{S}^{(m)} \int dx' \, D(x-x') \, [\bar{\psi}(x'), \gamma_\mu \psi(x')]^{(n-m)}. \tag{47}$$

The calculations given in the last two sections verify that the integration of our equations (1) and (2) is exactly equivalent to the solution of the equation of motion in the interaction representation[7]

$$i \frac{\delta \,|\, \psi[\sigma] >}{\delta \sigma(x)} = H(x) \,|\, \psi[\sigma] > . \tag{48}$$

(The notations are explained in the work by SCHWINGER.[7]) Our problem can, of course, be considered as solved, when we have obtained the expressions (32), (33), (26) and (43). In practical calculations, however, the iterated commutators give rise to rather lengthy computations, especially concerning the ε-functions. The P-symbols of DYSON are much more simple to handle and the ε-functions are there cause of considerably less trouble. Both methods, however, make use of the interaction Hamiltonian $H(x)$, and although this fact is no serious drawback as long as we are only considering particles of spin $\frac{1}{2}$, a quite different situation arises, when particles of integer spin are also treated. In this case the Hamiltonian contains terms, which are dependent of the surface σ (in the notations of SCHWINGER) and thus not truly invariant. As a matter of fact, these non-invariant terms will all drop out of the end formula but only after a long and comparatively complicated calculation.[8] The disappearance of the non-invariant terms is closely connected with the derivatives of the ε-functions, the occurrence of which in the formalisms of SCHWINGER and DYSON comes from the boundaries in the x_0-integrations. The integrals of equations (24) and (43) are effectively invariant only if the operators $H(x)$ commute on every space-like surface, and this is not the case if the interaction energy contains derivatives of the field operators. The mission of the non-invariant terms of $H(x)$ is only to counterbalance these inconvenient properties of the integrals. On the other hand, it has been shown in the paper referred to earlier[1] that, for the case of scalar particles in interaction with an electromagnetic field, it is possible to make calculations in the HEISENBERG representation and obtain invariant results directly without reference to any system of space-like surfaces. We therefore wish to solve our equations in a manner more similar to the procedure used in (I) and do not want to make use of the integral formulae above.

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[7] J. SCHWINGER, Phys. Rev. *74*, 1439 (1948).

5. Direct integration of the differential equations of quantum electrodynamics

As has been proved earlier[1], we can construct a solution to our equations in the following way

$$\psi^{(n+1)}(x) = -\frac{i}{2}\int S_R(x-x')\sum_{m=0}^{n}\{A_\nu^{(m)}(x'), \gamma_\nu\,\psi^{(n-m)}(x')\}\,d\,x' - \\ - \int S_R(x-x')\sum_{m=2}^{n+1}\delta\,m^{(m)}\,\psi^{(n+1-m)}(x')\,d\,x' \qquad (49)$$

$$A_\mu^{(n+1)}(x) = \frac{i}{2}\int D_R(x-x')\,\delta_{\mu\nu}\sum_{m=0}^{n}[\psi^{(m)}(x'), \gamma_\nu\,\psi^{(n-m)}(x')]\,d\,x'. \qquad (50)$$

Equations (49) and (50) permit us to compute the $n+1^{\text{st}}$ approximations of $\psi(x)$ and $A_\mu(x)$, when the operators $A_\mu^{(0)}(x)$, $\psi^{(0)}(x)$, $\ldots A_\mu^{(n)}(x)$, $\psi^{(n)}(x)$ are already known. The constants $\delta\,m^{(m)}$ have to be determined in such a way that all remaining divergences can be interpreted as charge-renormalization factors. The mass-renormalization is thus completely absorbed in the (infinite) quantity $\delta\,m$. As is easily shown[1] $\delta\,m^{(2n+1)}$ always vanish, and the summation in the last integral of (49) is effectively restricted to even integers m. If n is even, m takes the values 2, 4, ... n, and if n is odd, the corresponding set will be 2, 4, ... $n+1$. The solutions given above fulfil, from a formal point of view, the following boundary conditions

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[8] P. T. MATTHEWS, Phys. Rev. *76*, 1657 (1949), ibid. *76*, 684 (1949). Compare also R. JOST—J. RAYSKI, Helv. Phys. Acta *22*, 457 (1949) and G. KÄLLÉN, Helv. Phys. Acta *22*, 637 (1949).

$$\left.\begin{array}{l}\lim \psi^{(n)}(x) \to 0 \\ \lim A_\mu^{(n)}(x) \to 0\end{array}\right\} \begin{array}{l} x_0 \to -\infty \\ n \neq 0. \end{array} \qquad (51)$$

The incoming fields ($x_0 \to -\infty$) are thus described entirely by the terms $\psi^{(0)}(x)$ and $A_\mu^{(0)}(x)$. It is, however, also possible to construct other solutions to the differential equations, using either the advanced functions $S_A(x)$ and $D_A(x)$ or the expressions $\bar{S}(x)$ and $\bar{D}(x)$ instead of the retarded functions $S_R(x)$ and $D_R(x)$. In those cases the operators $\psi^{(0)}(x)$ and $A_\mu^{(0)}(x)$ will have another physical meaning. If the advanced functions are used, they will describe the outgoing fields ($x_0 \to +\infty$) instead of the incoming ones, and if the solutions are built up with the aid of the functions $\bar{S}(x)$ and $\bar{D}(x)$ the corresponding operators are given by half the sum of the incoming and the outgoing fields. As has been pointed out by YANG[6], both the incoming and the outgoing field operators will fulfil the usual commutation relations (equations (57) and (58) below) and are thus connected through a canonical transformation[9]

$$\psi^{(\text{out})}(x) = \boldsymbol{S}^{-1}\,\psi^{(\text{in})}(x)\,\boldsymbol{S}$$

$$A_\mu^{(\text{out})}(x) = \boldsymbol{S}^{-1}\,A_\mu^{(\text{in})}(x)\,\boldsymbol{S}.$$

The matrix $\boldsymbol{S}$ above is identical with the $\boldsymbol{S}$-matrix of HEISENBERG. If we use the singular functions $\bar{S}(x)$ and $\bar{D}(x)$, we get for the corresponding commutators

$$[A_\mu^{(0)}(x), A_\nu^{(0)}(x')] = \tfrac{1}{4}\left([A_\mu^{(\text{in})}(x), A_\nu^{(\text{in})}(x')] + [A_\mu^{(\text{out})}(x), A_\nu^{(\text{out})}(x')] + \right.$$
$$\left. + [A_\mu^{(\text{out})}(x), A_\nu^{(\text{in})}(x')] + [A_\mu^{(\text{in})}(x), A_\nu^{(\text{out})}(x')]\right). \qquad (52)$$

As $\boldsymbol{S}$ is unitary, half the *sum* of the two last terms in (52) is equal to the first term, and we obtain the same commutators for $A_\mu^{(0)}(x)$ as for $A_\mu^{(\text{in})}(x)$ and $A_\mu^{(\text{out})}(x)$. Hence, we also have

$$A_\mu^{(0)}(x) = \boldsymbol{Q}^{-1}\,A_\mu^{(\text{in})}(x)\,\boldsymbol{Q}.$$

A similar argument gives the corresponding connection between $\psi^{(0)}(x)$ and $\psi^{(\text{in})}(x)$. From the equations

$$A_\mu^{(0)}(x) = A_\mu^{(\text{in})}(x) + \frac{i}{4}\int D(x-x')\,[\bar{\psi}(x'), \gamma_\mu\,\psi(x')]\,dx'$$

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[9] Compare P. A. M. DIRAC, The Principles of Quantum Mechanics, third edition p. 106. (Oxford 1947).

$$A_\mu^{(\mathrm{out})}(x) = A_\mu^{(0)}(x) + \frac{i}{4}\int D(x-x')\,[\bar{\psi}(x'),\, \gamma_\mu \psi(x')]\,d x'$$

we infer, that we must also have

$$\begin{aligned} A_\mu^{(\mathrm{out})}(x) &= \boldsymbol{Q}^{-1} A_\mu^{(0)}(x)\, \boldsymbol{Q} \\ \psi^{(\mathrm{out})}(x) &= \boldsymbol{Q}^{-1} \psi^{(0)}(x)\, \boldsymbol{Q} \end{aligned} \tag{53 a}$$

with the same $\boldsymbol{Q}$ in (53) and (53 a). We thus obtain[10]

$$\boldsymbol{Q}^2 = \boldsymbol{S}. \tag{54}$$

If no real processes can take place, the $\boldsymbol{S}$-matrix and thus also the $\boldsymbol{Q}$-matrix are effectively equal to one, when we want to compute the expectation value of some operator. As long as we are only treating self-energy problems it is thus possible to simplify the solutions (49) and (50) in the following way

$$\psi^{(n+1)}(x) = -\frac{i}{2}\int \bar{S}(x-x') \sum_{m=0}^{n} \left\{A_\nu^{(m)}(x'),\, \gamma_\nu \psi^{(n-m)}(x')\right\} d x' - \int \bar{S}(x-x') \cdot \\ \cdot \sum_{m=2}^{n+1} \delta m^{(m)}\, \psi^{(n+1-m)}(x')\, d x' \tag{55}$$

$$A_\mu^{(n+1)}(x) = \frac{i}{2}\int \bar{D}(x-x')\,\delta_{\mu\nu} \sum_{m=0}^{n} [\bar{\psi}^{(m)}(x'),\, \gamma_\nu \psi^{(n-m)}(x')]\, d x'. \tag{56}$$

The operators $A_\mu^{(0)}(x)$ and $\psi^{(0)}(x)$ fulfil the following equations[1]

$$[A_\mu^{(0)}(x),\, A_\nu^{(0)}(x')] = -\,i\,\delta_{\mu\nu}\, D(x'-x) \tag{57}$$

$$\{\bar{\psi}_\alpha^{(0)}(x),\, \psi_\beta^{(0)}(x')\} = -\,i\, S_{\beta\alpha}(x'-x) \tag{58}$$

$$< \{A_\mu^{(0)}(x),\, A_\nu^{(0)}(x')\} >_0 = \delta_{\mu\nu}\, D^{(1)}(x'-x) \tag{59}$$

$$< [\bar{\psi}_\alpha^{(0)}(x),\, \psi_\beta^{(0)}(x')] >_0 = S_{\beta\alpha}^{(1)}(x'-x). \tag{60}$$

Equation (59) is actually incomplete as it stands but will always give correct results, if the theory is truly gauge-invariant.[11] Formulae (57) and (58) have already been used several times in the calculations above.

For what follows it is convenient to introduce the notations

$$\frac{1}{2^{n-1}}\left\{A_{\nu_1}^{(0)}(x'),\, \left\{\cdots\left\{A_{\nu_{n-1}}^{(0)}(x^{n-1})\; A_{\nu_n}^{(0)}(x^n)\right\}\cdots\right\}\right\} = A(1,\, 2,\, \ldots\, n). \tag{61}$$

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[10] Cf. DYSON[2], note to section V.

[11] Compare F. J. DYSON, Phys. Rev. 77, 420 (1950).

This symbol is actually symmetrical in all the variables x and indices ν. This can be inferred, if we consider the difference

$$\{A, \{B, C\}\} - \{B, \{A, C\}\} = [C, [B, A]] \tag{62}$$

where A, B and C are arbitrary operators. Putting

$$\begin{aligned} A &= A^{(0)}_{\nu_1}(x') \\ B &= A^{(0)}_{\nu_2}(x'') \\ C &= A(3, 4, \ldots n) \end{aligned}$$

we get

$$\begin{aligned} A(1, 2, 3, \ldots n) - A(2, 1, 3, \ldots n) &= \\ &= -i\,\delta_{\nu_1\nu_2}[A(3, \ldots n), D(x' - x'')] = 0. \end{aligned} \tag{63}$$

As $A(1, 2)$ is symmetrical in x' and x'', we conclude from (63) that $A(1, 2, 3)$ is completely symmetrical in all the variables x', x'' and x'''. From this fact the corresponding property of symmetry can be deduced for the iterated anticommutator of four potentials and so on for any value of n in (61).

The vacuum expectation value of the symbol, defined in (61), is easily computed according to well-known rules.[4] The operators have to be taken together in pairs in all possible ways, and the expectation value of every pair can then be calculated separately according to (59). If n in (61) is odd, the corresponding vacuum expectation value is obviously zero. If n is even, we obtain a sum of terms, one of which is given by

$$(\tfrac{1}{2})^{\frac{n}{2}}\,\delta_{\nu_1\nu_2}\,D^{(1)}(x' - x'')\,\delta_{\nu_3\nu_4}\,D(x''' - x'''') \ldots \delta_{\nu_{n-1}\nu_n}\,D(x^{n-1} - x^n). \tag{64}$$

The vacuum expectation value of the iterated anticommutator contains $(n-1)$ $(n-3) \ldots 5 \cdot 3$ terms of the form (64), one for each way, in which it is possible to combine the numbers $1, 2, \ldots n$ in pairs.

In our calculations, commutators and anticommutators of the operators (61) will frequently appear. These quantities can be calculated successively in the following way. We begin by showing that

$$\tfrac{1}{2}[A^{(0)}_{\nu_1}(x'), A(2, 3, \ldots n)] = -\frac{i}{2}\sum_{m=2}^{n} \delta_{\nu_1\nu_m}\,D(x^m - x')\,A^{(m)}(2, 3, \ldots n) \tag{65}$$

where

$$A^{(m)}(1, 2, \ldots n) = A(1, 2, \ldots m-1, m+1, \ldots n). \tag{66}$$

Supposing (65) to be valid for some value n, we have

$$\begin{aligned}
\tfrac{1}{2}\left[A_{\nu_1}^{(0)}(x'),\, A(2,3,\ldots n+1)\right] &= \tfrac{1}{4}\left\{A_{\nu_2}^{(0)}(x''),\, \left[A_{\nu_1}^{(0)}(x'),\, A(3,\ldots n+1)\right]\right\} + \\
&+ \tfrac{1}{4}\left\{A(3,\ldots n+1)\left[A_{\nu_1}^{(0)}(x'),\, A_{\nu_2}^{(0)}(x'')\right]\right\} = \\
&= -\frac{i}{4}\sum_{m=3}^{n+1}\left\{A_{\nu_2}^{(0)}(x''),\, A^{(m)}(3,\ldots n+1)\right\}\cdot\delta_{\nu_1\nu_m}\cdot \\
&\cdot D(x^m - x') - \frac{i}{2}A(3,\ldots n+1)\,\delta_{\nu_1\nu_2}D(x''-x') = \\
&= -\frac{i}{2}\sum_{m=2}^{n+1}\delta_{\nu_1\nu_m}D(x^m - x')\,A^{(m)}(2,\ldots n+1). \qquad (67)
\end{aligned}$$

As (65) is certainly correct for $n = 1$, it is thus proved for all values of n.

Using this result we are now able to compute

$$\begin{aligned}
\tfrac{1}{2}\left\{A(1,2),\, A(3,4,\ldots n)\right\} &= A(1,2,\ldots n) + \\
&+ \tfrac{1}{4}\left[A_{\nu_2}^{(0)}(x''),\, \left[A_{\nu_1}^{(0)}(x'),\, A(3,4,\ldots n)\right]\right] = A(1,2,\ldots n) + \\
&+ \left(-\frac{i}{2}\right)^2\sum_{m+k=3}^{n}\delta_{\nu_1\nu_m}D(x^m-x')\,\delta_{\nu_2\nu_k}D(x^k-x'')\,A^{(m)(k)}(2,3,\ldots n). \qquad (68)
\end{aligned}$$

Continuing in this way and calculating alternatively the commutators and the anticommutators, we obtain the following formulae

$$\begin{aligned}
\tfrac{1}{2}\left\{A(1,2,\ldots m),\, A(m+1,\ldots n)\right\} &= A(1,2,\ldots n) + \\
&+ \left(-\frac{i}{2}\right)^2\sum_{i<j=1}^{m}\ \sum_{\substack{r+s=\\=m+1}}^{n}\delta_{\nu_i\nu_r}D(x^r-x^i)\cdot\delta_{\nu_j\nu_s}D(x^s-x^j)\cdot \\
&\cdot A^{(i)(j)(r)(s)}(1,2,\ldots n) + \left(-\frac{i}{2}\right)^4\sum_{\substack{i<j<k<\\<l=1}}^{m}\ \sum_{\substack{r+s+t+\\+u=m+1}}^{n}\delta_{\nu_i\nu_r}D(x^r-x^i)\cdot \\
&\cdot\delta_{\nu_j\nu_s}D(x^s-x^j)\,\delta_{\nu_k\nu_t}D(x^t-x^k)\,\delta_{\nu_u\nu_l}D(x^u-x^l)\cdot \\
&\cdot A^{(i)(j)(k)(l)(r)(s)(t)(u)}(1,2,3\ldots n) + \left(-\frac{i}{2}\right)^6\sum\sum\cdots+\cdots \qquad (69)
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{1}{2}\left[A(1,2,\ldots m),\, A(m+1,\ldots n)\right] &= -\frac{i}{2}\sum_{i=1}^{m}\ \sum_{r=m+1}^{n}\delta_{\nu_i\nu_r}D(x^r-x^i)\cdot \\
&\cdot A^{(i)(r)}(1,2,\ldots n) + \left(-\frac{i}{2}\right)^3\sum_{i<j<k=1}^{m}\ \sum_{\substack{r+s+t=\\=m+1}}^{n}\delta_{\nu_i\nu_r}D(x^r-x^i)\,\delta_{\nu_j\nu_s}D(x^s-x^j)\cdot \\
&\cdot\delta_{\nu_k\nu_t}D(x^t-x^k)\,A^{(i)(j)(k)(r)(s)(t)}(1,2,\ldots n) + \left(-\frac{i}{2}\right)^5\sum\sum\cdots+\cdots. \qquad (70)
\end{aligned}$$

The summations over $i, j, \ldots$ and $r, s, \ldots$ have to be done in such a way, that all possible combinations of the differences $x^{\xi} - x^{\eta}$, where η belongs to the set $i, j, \ldots$ and ξ to $r, s, \ldots$, appear once and only once.

For the operators $\psi^{(0)}(x)$ and $\bar{\psi}^{(0)}(x)$ the definitions similar to (61) will be

$$J(1, 2, \ldots i_1; i_1 + 1, \ldots i_2; \ldots \quad \ldots i_r; i_r + 1, \ldots n) =$$

$$= \frac{1}{2^{2r+1}} [\bar{\psi}^{(0)}(x'), \bar{S}(1, 2, \ldots i_1) \{\psi^{(0)}(x^{i_1}), [\bar{\psi}^{(0)}(x^{i_1+1}),$$

$$\bar{S}(i_1 + 1, \ldots i_2) \{\psi^{(0)}(x^{i_2}), [\ldots \quad \ldots [\bar{\psi}^{(0)}(x^{i_r+1}),$$

$$\bar{S}(i_r + 1, \ldots n) \psi^{(0)}(x^n)] \ldots]\}]\}] \tag{71}$$

where

$$\bar{S}(1, 2, \ldots n) = \gamma_{\nu_1} \bar{S}(x' - x'') \gamma_{\nu_2} \ldots \gamma_{\nu_{n-1}} \bar{S}(x^{n-1} - x^n) \gamma_{\nu_n}. \tag{72}$$

The vacuum expectation value of (71) is easily found to be a sum of $(r + 1)!$ terms, one typical of which can be written

$$\frac{(-1)^r}{2^{r+1}} Sp [S^{(1)}(x^n - x') \bar{S}(1, 2, \ldots i_1) S^{(1)}(x^{i_1} - x^{i_1+1}) \bar{S}(i_1 + 1, \ldots i_2) \ldots$$

$$\ldots S^{(1)}(x^{i_r} - x^{i_r+1}) \bar{S}(i_r + 1, \ldots n)]. \tag{73}$$

The other terms are obtained, if the field operators are combined in different ways and can contain a product of two or more traces. The numerical value of the factor in front of the expression is always 2^{-r-1} ($r + 1$ is the number of $S^{(1)}$-functions in the expectation value) but the sign will be dependent on the number of traces in the product. As a careful inspection shows, the sign is given by

$$(-1)^{r-l+1} \tag{74}$$

where l is the number of factors in our expression. For the commutators and anticommutators between two J-symbols we get formulae similar to (69) and (70). Arguing in the same way as before, we obtain *e.g.*

$$\tfrac{1}{2} \{J(1, 2, \ldots m), J(m+1, \ldots i_1; i_1+1, \ldots i_2; \ldots \quad \ldots i_r; i_r+1, \ldots n)\} =$$

$$= J(1, 2, \ldots m; m + 1, \ldots i_1; \ldots \quad \ldots i_r; i_r + 1, \ldots n) +$$

$$+ \left(-\frac{i}{2}\right)^2 \sum_{\alpha, \beta} J(m + 1, \ldots i_1; \ldots$$

$$\ldots i_\alpha; i_\alpha + 1; \ldots \boldsymbol{i}_{\alpha+1}; \mathbf{1}, 2, \ldots \boldsymbol{m}; \boldsymbol{i}_\beta + \mathbf{1}, \ldots \quad \ldots i_r; i_r + 1, \ldots n) \tag{75}$$

where

$$J(1, 2, \ldots i_1; i_1 + 1, \ldots \quad \ldots \boldsymbol{i_\alpha}; \boldsymbol{i_\alpha} + \mathbf{1}, \ldots \boldsymbol{i_\beta}; \boldsymbol{i_\beta} + \mathbf{1}, \ldots$$

$$\ldots i_r; i_r + 1, \ldots n) = \frac{1}{2^{2r-1}} [\bar{\psi}^{(0)}(x'), \bar{S}(1, 2, \ldots i_1) \{\psi^{(0)}(x^{i_1}),$$

$$[\cdots \quad \cdots \{\psi^{(0)}(x^{i_{\alpha-1}}), [\bar{\psi}^{(0)}(x^{i_{\alpha-1}+1}), \bar{S}(i_{\alpha-1} + 1, \ldots i_\alpha) S(x^{i_\alpha} - x^{i_\alpha+1}) \cdot$$

$$\cdot \bar{S}(i_\alpha + 1, \ldots i_\beta) S(x^{i_\beta} - x^{i_\beta+1}) \bar{S}(i_\beta + 1, \ldots i_{\beta+1}) \{\psi^{(0)}(x^{i_{\beta+1}}),$$

$$[\cdots \quad \cdots [\bar{\psi}^{(0)}(x^{i_r+1}), \bar{S}(i_r + 1, \ldots n) \psi^{(0)}(x^n)] \cdots]\}]\} \cdots]\}]. \tag{76}$$

Equation (76) corresponds to (68), and the appearance of the two $S(x)$-functions in (76) is quite analogous to the existence the $D(x)$-functions in (68). One important difference between the two formulae, however, lies in the fact, that the summation in the last term of (75) has to include also the possibilities $\alpha = \beta$. In these cases a trace of the form

$$Sp[S(x^{i_{\alpha+1}} - x') \bar{S}(1, 2, \ldots n) S(x^n - x^{i_\alpha+1}) \bar{S}(i_\alpha + 1, \ldots i_{\alpha+1})] \tag{77}$$

has to be substituted in (76) in the place of the expression containing the $S(x)$-functions. This difference between (75) and (68) has its origin in the fact, that the symbols J contain two kinds of operators ($\bar{\psi}^{(0)}(x)$ and $\psi^{(0)}(x)$), while the expressions A in (61) are built up from only one type of electromagnetic potentials.

We will make no attempt at writing down explicitly the commutators and the other, more general anticommutators between the J-symbols, as they will be only rather trivial modifications of the formulae (69) and (70) but will nevertheless give rise to comparatively complicated notations. The main structure of those expressions is, however, easily inferred from what has already been said.

We are now able to turn to the direct solution of the equations (1) and (2). As we do not want to introduce too many complications at the beginning, for the moment we drop the mass-renormalization terms and write (55) and (56) as

$$\psi^{(n+1)}(x) = -\frac{i}{2} \int \bar{S}(x - x') \sum_{m=0}^{n} \{A_\nu^{(m)}(x'), \gamma_\nu \psi^{(n-m)}(x')\} \, dx' \tag{78}$$

$$A_\mu^{(n+1)}(x) = \frac{i}{2} \int \bar{D}(x - x') \sum_{m=0}^{n} [\bar{\psi}^{(m)}(x'), \gamma_\mu \psi^{(n-m)}(x')] \, dx'. \tag{79}$$

From these equations we get successively

$$\psi^{(1)}(x) = -i \int \bar{S}(x - x') \gamma A^{(0)}(x') \psi^{(0)}(x') \, dx' \tag{80 a}$$

$$A_\mu^{(1)}(x) = \frac{i}{2} \int \bar{D}(x - x') [\bar{\psi}^{(0)}(x'), \gamma_\mu \psi^{(0)}(x')] \, dx' \tag{80 b}$$

$$\begin{aligned}\psi^{(2)}(x) = (-i)^2 \iint \bar{S}(x-x')\,[\bar{S}(1,2)\,\psi^{(0)}(x'')\,A(1,2) - \tfrac{1}{4}\gamma_{\nu_1}\{\psi^{(0)}(x'),\\ [\bar{\psi}^{(0)}(x''),\gamma_{\nu_2}\psi^{(0)}(x'')]\}\,\delta_{\nu_1\nu_2}\bar{D}(x'-x'')]\,dx'\,dx''\end{aligned} \tag{81 a}$$

$$A_\mu^{(2)}(x) = -i^2 \iint J(1,2)\,[\delta_{\mu\nu_1}\bar{D}(x-x')\,A_{\nu_2}^{(0)}(x'') + \delta_{\mu\nu_2}\bar{D}(x-x'')\cdot A_{\nu_1}^{(0)}(x')] \tag{81 b}$$

$$\begin{aligned}\psi^{(3)}(x) = (-i)^3 \iiint \bar{S}(x-x')\,[\bar{S}(1,2,3)\,\psi^{(0)}(x''')\,A(1,2,3) - \tfrac{1}{2}\bar{S}(1,2)\cdot\\ \cdot\{\psi^{(0)}(x''),J(3)\}\,(A_{\nu_1}^{(0)}(x')\,\delta_{\nu_2\nu_3}\bar{D}(x''-x''') + A_{\nu_2}^{(0)}(x'')\,\delta_{\nu_1\nu_3}\bar{D}(x'-x''')) -\\ -\tfrac{1}{2}\gamma_{\nu_1}\{\psi^{(0)}(x'),J(2,3)\}\,(A_{\nu_2}^{(0)}(x'')\,\delta_{\nu_1\nu_3}\bar{D}(x'-x''') + A_{\nu_3}^{(0)}(x''')\,\delta_{\nu_1\nu_2}\cdot\\ \cdot\bar{D}(x'-x''))]\,dx'\,dx''\,dx'''\end{aligned} \tag{82 a}$$

$$\begin{aligned}A_\mu^{(3)}(x) = i^3 \iiint \{J(1,2,3)\,(\delta_{\mu\nu_1}\bar{D}(x-x')\,A(2,3) + \delta_{\mu\nu_2}\bar{D}(x-x'')\,A(1,3) +\\ +\delta_{\mu\nu_3}\bar{D}(x-x''')\,A(1,2)) - J(1,2;3)\,(\delta_{\mu\nu_1}\bar{D}(x-x')\,\delta_{\nu_2\nu_3}\bar{D}(x''-x''') +\\ +\delta_{\mu\nu_2}\bar{D}(x-x'')\,\delta_{\nu_1\nu_3}\bar{D}(x'-x'''))\}\,dx'\,dx''\,dx''' + \frac{i}{4}\iiint \bar{D}(x-x')\cdot\\ \cdot\delta_{\mu\nu_1}Sp\,[S(x''-x''')\,\bar{S}(3,2,1)]\,\delta_{\nu_2\nu_3}D(x''-x''')\,dx'\,dx''\,dx'''.\end{aligned} \tag{82 b}$$

The last term in (82 b) is obtained from the commutator between $\bar{\psi}^{(1)}(x)$ and $\psi^{(1)}(x)$. This expression gives

$$\begin{aligned}\tfrac{1}{2}[\bar{\psi}^{(1)}(x'),\gamma_{\nu_1}\psi^{(1)}(x')] = \frac{(-i)^2}{4}\iint dx''\,dx'''\,\{[\bar{\psi}^{(0)}(x'''),\bar{S}(3,1,2)\,\psi^{(0)}(x'')]\cdot\\ \cdot 2A(3,2) + \{\bar{\psi}^{(0)}(x'''),\bar{S}(3,1,2)\,\psi^{(0)}(x'')\}\cdot[A_{\nu_3}^{(0)}(x'''),A_{\nu_2}^{(0)}(x'')]\}.\end{aligned} \tag{83}$$

The first term in (83) is equal to

$$-\iint dx''\,dx'''\,J(3,1,2)\,A(3,2) \tag{84}$$

and the second one gives with the aid of (57) and (58)

$$\tfrac{1}{4}\iint dx''\,dx'''\,Sp\,[S(x''-x''')\,\bar{S}(3,1,2)]\,\delta_{\nu_2\nu_3}D(x''-x'''). \tag{85}$$

Up till now the formulae obtained have contained only the singular functions $\bar{S}(x)$ and $\bar{D}(x)$, but in (82 b) and (85) also the functions $S(x)$ and $D(x)$ have appeared. When we proceed to still higher approximations, more and more terms of this form will turn up from the commutators and anticommutators in (78) and (79). Computing, as a simple example, the commutator

$$\begin{aligned}\tfrac{1}{2}[\bar{\psi}^{(0)}(x')\,\bar{S}(1,2,\ldots m)\,A(1,2,\ldots m),\\ A(m+1,\ldots n)\,\bar{S}(m+1,\ldots n)\,\psi^{(0)}(x^n)]\end{aligned} \tag{86}$$

we obtain

$$
\begin{aligned}
& J(1, 2, \ldots n)\, \tfrac{1}{2} \{A(1, 2, \ldots m), A(m+1, \ldots n)\} - \\
& \quad - \frac{i}{4} Sp\,[S(x^n - x')\, \bar{S}(1, 2, \ldots n)] \cdot [A(1, 2, \ldots m), \\
& \quad A(m+1, \ldots n)] = J(1, 2, \ldots n)\, [A(1, 2, \ldots n) + \\
& \quad + \left(-\frac{i}{2}\right)^2 \sum_{i<j=1}^{m} \sum_{\substack{r+s= \\ =m+1}}^{n} \delta_{\nu_i \nu_r} D(x^r - x^i)\, \delta_{\nu_j \nu_s} D(x^s - x^j) \cdot \\
& \quad A^{(i)(j)(r)(s)}(1, 2, \ldots n) + \cdots] + Sp\,[S(x^n - x')\, \bar{S}(1, 2, \ldots n)] \cdot \\
& \quad \cdot \left[\left(-\frac{i}{2}\right)^2 \sum_{i=1}^{m} \sum_{r=m+1}^{n} \delta_{\nu_i \nu_r} D(x^r - x^i)\, A^{(i)(r)}(1, 2, \ldots n) + \cdots\right]. \qquad (87)
\end{aligned}
$$

If an expectation value of the first term is calculated, we obtain, according to the rules given above, a sum of terms, each of which contains only $\bar{S}(x)$, $S^{(1)}(x)$, $\bar{D}(x)$ and $D^{(1)}(x)$ functions. The variables of those singular functions are always differences between two coordinates $x^i - x^j$. If some $j < m$ and at the same time the attached $i > m$, there is a simple correspondence between these terms and the terms obtained from the expectation value of the remaining expression in (87). The latter are obtained from the former, if we in every term substitute two, four, six etc. "unbarred functions" for the $D^{(1)}$ and $S^{(1)}$ functions in all possible ways, only remembering the conditions for i and j. The same correspondence is easily seen to hold even if the more general expressions

$$\tfrac{1}{2} \bar{S}(1, \ldots n) \{\psi^{(0)}(x^n), J(n+1, \ldots; \ldots \ \ldots)\} \qquad (88)$$

are used in (86) instead of $\bar{S}(1, \ldots n)\, \psi^{(0)}(x^n)$.

It is now convenient to split our operators $\psi^{(n)}(x)$ and $A_\mu^{(n)}(x)$ into two parts, one of which (${}^{(1)}\psi^{(n)}(x)$ and ${}^{(1)}A_\mu^{(n)}(x)$) contains no "unbarred functions", and write

$$\psi^{(n)}(x) = {}^{(1)}\psi^{(n)}(x) + {}^{(2)}\psi^{(n)}(x) \qquad (89\text{ a})$$

$$A_\mu^{(n)}(x) = {}^{(1)}A_\mu^{(n)}(x) + {}^{(2)}A_\mu^{(n)}(x). \qquad (89\text{ b})$$

From the definitions given earlier we immediately infer

$$
\begin{aligned}
& {}^{(1)}\psi^{(n)}(x) = (-i)^n \int \cdots \int dx' \ldots dx^n\, \tfrac{1}{2} \bar{S}(x - x') \sum_{r} \sum_{i_1 \ldots i_r} \bar{S}(1, 2, \ldots i_1) \cdot \\
& \quad \cdot \{\psi^{(0)}(x^{i_1}), J(i_1 + 1, \ldots i_2; i_2 + 1, \ldots; \ldots \ \ldots i_r; i_r + 1, \ldots n)\, (-1)^r \cdot \\
& \quad \cdot \sum_{\alpha_1 \ldots \alpha_{2r}} \delta_{\nu_{\alpha_1} \nu_{\alpha_2}} \bar{D}(x^{\alpha_1} - x^{\alpha_2})\, \delta_{\nu_{\alpha_3} \nu_{\alpha_4}} \bar{D}(x^{\alpha_3} - x^{\alpha_4}) \ldots \delta_{\nu_{\alpha_{2r-1}} \nu_{\alpha_{2r}}} \cdot \\
& \quad \cdot \bar{D}(x^{\alpha_{2r-1}} - x^{\alpha_{2r}})\, A^{(\alpha_1) \ldots (\alpha_{2r})}(1, 2, \ldots n). \qquad (90)
\end{aligned}
$$

The summation over r has to be performed from 0 to $\frac{n}{2}$, if n is even, and from 0 to $\frac{n-1}{2}$ if n is odd. The numbers $i_1, \ldots i_r$ have to be summed over all possible combinations for a fixed value of r. The last sum over the α_j has to include all possibilities, where each one of the sets $1, \ldots i_1$; $i_1+1, \ldots i_2$; … …; $i_r+1, \ldots n$ contains at least one of the numbers $\alpha_1 \ldots \alpha_{2r}$. The summations over α_j are also restricted through another condition, which can be formulated in the following way. Consider one of the sets $i_j+1, \ldots i_{j+1}$. To this set belongs at least one value α_j and a function $\bar{D}(x^{\alpha_j} - x^{\alpha_k})$, where the number α_k is included in another set $i_k+1, \ldots i_{k+1}$. We then say, that the two sets $i_j+1, \ldots i_{j+1}$ and $i_k+1, \ldots i_{k+1}$ are "connected". If a third set is now connected, in the sense just defined, to one of the two mentioned earlier, we say that it is also connected to the other one etc. The remaining restriction for the α-summations can then be so formulated, that none of the sets $i_j+1, \ldots i_{j+1}$ is to be connected to itself. As an example of a term, which does *not* appear according to this restriction, we can mention

$$-\tfrac{1}{2}(-i)^3 \iiint \bar{S}(x-x')\,\gamma_{\nu_1}\{\psi^{(0)}(x'), J(2,3)\}\cdot$$
$$.\,A^{(0)}_{\nu_1}(x')\,\delta_{\nu_2\nu_3}\,\bar{D}(x''-x''')\,dx'\,dx''\,dx'''.$$

(Compare equation (82 a)).

The corresponding formula for the electromagnetic potentials is given by

$$\begin{aligned}{}^{(1)}A^{(m)}_{\mu}(x) = (-i)^n \int\cdots\int dx' \ldots dx^n \sum_{r}\sum_{i_1\ldots i_r} J(1,2,\ldots i_1; i_1+1, \ldots\\ \ldots i_r; i_r+1, \ldots n)\,(-1)^{r+1} \sum_{\alpha_1\ldots\alpha_{2r}} \delta_{\nu_{\alpha_1}\nu_{\alpha_2}}\bar{D}(x^{\alpha_1}-x^{\alpha_2}) \ldots \delta_{\nu_{\alpha_{2r-1}}\nu_{\alpha_{2r}}}\cdot\\ \cdot\bar{D}(x^{\alpha_{2r-1}}-x^{\alpha_{2r}}) \sum_{j}{}' \delta_{\mu\nu_j}\bar{D}(x-x^j)\,A^{(j)(\alpha_1)\ldots(\alpha_{2r})}(1,2,\ldots n).\end{aligned} \tag{91}$$

The same conventions for the summations over r, $i_1, \ldots i_r$, $\alpha_1, \ldots \alpha_{2r}$ are valid in (91) and (92). The index j in (91) has to be summed over all values from 0 to n, excepting the numbers $\alpha_1, \ldots \alpha_{2r}$. The proof of (90) and (91) is conveniently done from n to $n+1$ in the manner used several times earlier and will not be given explicitly. In this calculation it is permitted to disregard all terms in the right-hand side of (69) and (70) except the first one in (69), as those terms will give contributions only to ${}^{(2)}\psi^{(n)}(x)$ and ${}^{(2)}A^{(n)}_{\mu}(x)$.

The remaining parts of the operators $({}^{(2)}\psi^{(n)}\,x$ and ${}^{(2)}A^{(n)}_{\mu}(x))$ cannot be written down in such a compact form, as was possible for ${}^{(1)}\psi^{(n)}(x)$ and ${}^{(1)}A^{(n)}_{\mu}(x)$. Their general properties, however, can easily be inferred from the remark made after equation (87). As these expressions will all appear from commutators of the form (86), it is very easy to include, later on, their contributions to the expectation values of our operators. We thus turn to the calculation of expectation values and choose for this purpose the current operator as a suitable expression. This operator has a fundamental significance as both the interaction Hamiltonian and the **S**-matrix are closely connected with it. If we take only the part ${}^{(1)}j(x)$ defined analogously to (89), we obtain with the definition

$$ {}^{(1)}j_\mu(x) = \sum_{n=0}^{\infty} e^{n+1}\, {}^{(1)}j_\mu^{(n)}(x) \tag{92} $$

the formula

$$ \begin{aligned} {}^{(1)}j_\mu^{(n)}(x) = (-i)^n \int\cdots\int d\,x' \ldots d\,x^n \sum_{m=0}^{n}{}' \sum_{r} \sum_{i_1 \ldots i_r} (-i)^r J\,(1,\, 2,\, \ldots\, i_1;\, \ldots \\ \ldots\, i_s;\, i_s + 1,\, \ldots\, m,\, 0,\, m+1,\, \ldots\, i_{s+1};\, \ldots \quad \ldots\, i_r;\, i_r + 1,\, \ldots\, n) \cdot \\ \cdot \sum_{\alpha_1 \ldots \alpha_{2r}} \delta_{\nu_{\alpha_1} \nu_{\alpha_2}} \bar{D}\,(x^{\alpha_1} - x^{\alpha_2}) \ldots \delta_{\nu_{\alpha_{2r-1}} \nu_{\alpha_{2r}}} \bar{D}\,(x^{\alpha_{2r-1}} - x^{\alpha_{2r}}) \cdot \\ \cdot A^{(\alpha_1) \ldots (\alpha_{2r})}(1,\, 2,\, \ldots\, n). \end{aligned} \tag{93} $$

The summation conventions for $r, \ldots \alpha_{2r}$ are similar to the earlier ones. The expectation value is now obtained as a sum of several terms, each of them containing $\frac{n}{2}$ functions $S^{(1)}(x)$ or $D^{(1)}(x)$. It is at this point most convenient to introduce the graphical language of DYSON[2, 3] and FEYNMAN.[5] If our expression contains $n+1$ variables $x, x', \ldots x^n$, we represent it by $n+1$ points in a graph, each point labelled with one of the letters $x, x', \ldots x^n$. For every function $\bar{S}(x^i - x^j)$ or $S^{(1)}(x^i - x^j)$ in the expectation value we draw a line (an electron line) from the point x^i to the point x^j, and for every function $\bar{D}(x^i - x^j)$ or $D^{(1)}(x^i - x^j)$ we draw a dotted line (a photon line) connecting the two points in question. For every operator representing a free particle we also draw a line (electron or photon) from the edge of our graph to the point that appears as a variable in the operator. In this way we obviously obtain a simple diagram representing our expectation value. It is, however, doubtful, if this graph will also give a picture of the physical process underlying the theory. In the older formulation of quantum electrodynamics the higher radiative approximations were described with the aid of virtual photons and virtual pairs of electrons. In the papers quoted above, FEYNMAN[5] has developed an intuitive method of identifying the electron lines with the virtual pairs and the photon lines with the virtual quanta. From the point of view adopted here, this is, however, only an extra complication, as we have no need for the conceptions of virtual particles, and as the singular functions have the much more natural property of being either solutions to the inhomogenous differential equations or expectation values. Also in the older formulation of quantum electrodynamics the virtual particles were, even from a principal point of view, not observable and thus only a mathematical conception. We therefore prefer to regard our graphs only as a suitable representation of the mathematical structure of the theory.

It might be pointed out here, that it is also possible and in some respects convenient to represent the field operators themselves with the aid of graphical figures. If we use the same conventions as above concerning the $\bar{S}(x^i - x^j)$ and $\bar{D}(x^i - x^j)$ functions but write a letter ψ, ψ or A for every operator $\psi^{(0)}(x^i)$, $\psi^{(0)}(x^i)$ and $A_\mu^{(0)}(x^i)$, we can make us a diagram of the ${}^{(1)}\psi(x)$ and ${}^{(1)}A_\mu(x)$ in a simple way. For the operators given by (80), (81) and (82) we then obtain figures 5, 6, 7 and 8. The summation rules of equation (90) can here be expressed in the following way. We sum over all possible combinations of either n operators A and one operator ψ or $(n-2)$ operators A, two operators ψ and one operator ψ or $(n-4)$ operators A, three operators ψ and two operators $\bar{\psi}$ etc., where the different operators are connected together with electron and photon lines. The restrictions on the α_i-variables correspond to the restriction to only "connected" graphs with no "closed loops".[12] In a similar way the ${}^{(1)}j_\mu^{(n)}(x)$ operators (not their expectation values) can be represented by graphs containing ψ and A as above. In the lowest approximations we obtain, corresponding to equation (93), figures 9 and 10.

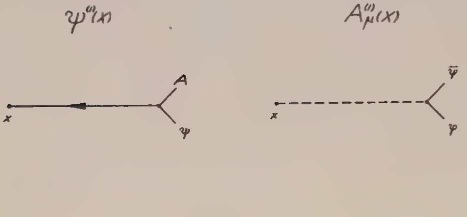

Fig. 5.

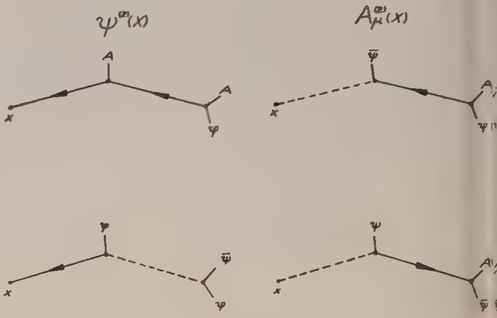

Fig. 6.

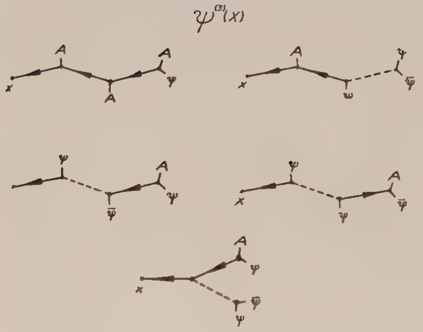

Fig. 7.

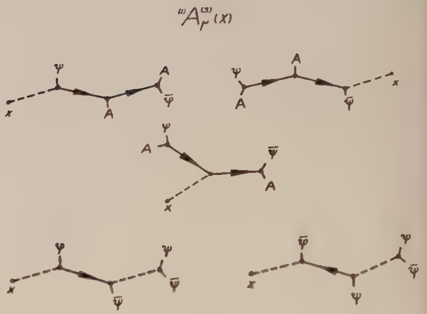

Fig. 8.

We obtain the graphs for the expectation values from the graphs for the operators by combining the letters A (or ψ and ψ) together in pairs and sub-

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[12] For the definition of these expressions see DYSON[2] p. 495—496.

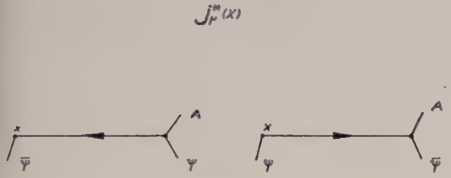

Fig. 9.

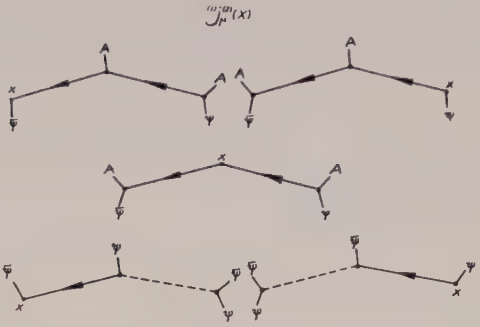

Fig. 10.

stituting a photon (electron) line for each pair. This procedure has to be performed in all possible ways, and it is easily seen, that all the connected graphs of DYSON[12] will appear as a result. No disconnected graph, however, will turn up, but these diagrams were shown by DYSON to give only an unimportant vacuum fluctuation and no contribution to the expectation value.[12] It might rather be considered as being to the advantage of our formalism, that these terms will not come out from the calculation.

For each connected graph of DYSON we will get, not only one term as he does, but several terms containing both the $S^{(1)}(x)$, $D^{(1)}(x)$ and $\bar S(x)$, $\bar D(x)$ functions. If we consider a graph belonging to the n^{th} approximation, this diagram does necessarily contain exactly $\dfrac{n}{2}$ closed loops, which may or may not have one or several lines in common. If we first consider the case, when all the loops are entirely separated from each other, we infer from the rules given above that the lines joining the loops must be represented by $\bar S(x)$ or $\bar D(x)$ functions. This follows from the fact, that the diagrams of the operators (*e.g.* figures 9 and 10) contain no disconnected parts. As a simple example we

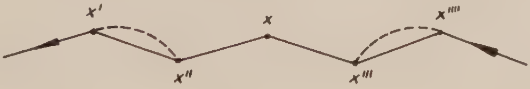

Fig. 11.

can consider the graph in figure 11, belonging to the one-particle part of $^{(1)}j_\mu^{(4)}(x)$. This diagram contains two loops, one between (x', x'') and the other between (x''', x''''). The lines joining the loops are the lines between the points (x'', x) and (x, x'''). These two last lines are thus represented by the functions $\bar S(x'' - x)$ and $\bar S(x - x''')$. As the diagrams of the operators contain no closed loops, one and only one of the lines in every loop in the graph of an expectation value must be represented by a $S^{(1)}(x)$ or $D^{(1)}(x)$ function. It is also obvious, that all possible combinations will actually appear, and this means, that each loop is represented by a *factor* of the following general form

$$\bar{S}(1, 2, \ldots n)\, \delta_{\nu_n \nu_1} D^{(1)}(x^n - x') + \sum_{m=1}^{n-1} \bar{S}(1, 2, \ldots m)\, S^{(1)}(x^m - x^{m+1}) \cdot$$
$$\cdot \bar{S}(m + 1, \ldots n) \cdot \delta_{\nu_1 \nu_n} \bar{D}(x^n - x'). \tag{94}$$

The modifications necessary, when more or less than one of the lines are photon lines, are obvious. For the special graph of figure 11 we thus obtain, apart from a numerical factor

$$\int \cdots \int d x' \ldots d x'''' \, \bar{\psi}^{(0)}(x')\, \gamma_\lambda \{\bar{S}(x' - x'')\, D^{(1)}(x'' - x') +$$
$$+ S^{(1)}(x' - x'')\, \bar{D}(x'' - x')\} \cdot \gamma_\lambda \bar{S}(x'' - x)\, \gamma_\mu \bar{S}(x - x''')\, \gamma_\varrho \cdot$$
$$\cdot \{\bar{S}(x''' - x'''')\, D^{(1)}(x'''' - x''') + S^{(1)}(x''' - x'''')\, \bar{D}(x'''' - x''')\} \cdot$$
$$\cdot \gamma_\varrho \psi^{(0)}(x''''). \tag{95}$$

Using the following integral representation, given by LUTTINGER[13]

$$\frac{\delta(a_1)}{a_2 \ldots a_n} + \frac{\delta(a_2)}{a_1 a_3 \ldots a_n} + \cdots + \frac{\delta(a_n)}{a_1 \ldots a_{n-1}} =$$
$$= (-1)^{n-1} \int_0^1 d x_1 \int_0^{x_1} d x_2 \ldots \int_0^{x_{n-2}} d x_{n-1}\, \delta^{(n-1)}(a_n x_{n-1} +$$
$$+ a_{n-1}(x_{n-2} - x_{n-1}) + \cdots + a_2(x_1 - x_2) + a_1(1 - x_1)) \tag{96}$$

(the proof of equation (96) is conveniently done from n to $n+1$) we can, in momentum space, write the expression (94) as

$$\left(\frac{1}{2\pi}\right)^{4n-1} \int \cdots \int d p_1 \ldots d p_{n-1}\, d k\, e^{i p_1(x' - x'') + i p_2(x'' - x''') + \cdots + i p_{n-1}(x^{n-1} - x^n)} \cdot$$
$$\cdot e^{i k (x^n - x')} \cdot \gamma_{\nu_1}(i \gamma p_1 - m)\, \gamma_{\nu_2} \ldots \gamma_{\nu_{n-1}}(i \gamma p_{n-1} - m)\, \gamma_{\nu_n} \cdot$$
$$\cdot \delta_{\nu_1 \nu_n} (-1)^{n-1} \int_0^1 d \xi_1 \ldots \int_0^{\xi_{n-2}} d \xi_{n-1} \cdot \delta^{(n-1)}(k^2 \xi_{n-1} + p_{n-1}^2(\xi_{n-2} - \xi_{n-1}) +$$
$$+ \cdots + p_2^2(\xi_1 - \xi_2) + p_1^2(1 - \xi_1) + m^2(1 - \xi_{n-1})). \tag{97}$$

For the special case of equation (95) we obtain in this way

$$\left(\frac{1}{2\pi}\right)^6 \int \int d k_1\, d k_2 \int \cdots \int d p_1 \ldots d p_4\, e^{i x (p_3 - p_2)}\, \bar{u}(p_1 - k_1)\, \gamma_{\lambda_1}(i \gamma p_1 - m) \cdot$$
$$\cdot \gamma_{\lambda_1}(i \gamma p_2 - m) \cdot \gamma_\mu (i \gamma p_3 - m)\, \gamma_{\lambda_2}(i \gamma p_4 - m)\, \gamma_{\lambda_2} u(p_4 - k_2) \cdot$$

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[13] J. M. LUTTINGER, unpublished letter to the author.

$$\cdot \frac{\delta(k_1 - p_1 + p_2)\,\delta(p_4 - p_3 - k_2)}{(p_2^2 + m^2)\ (p_3^2 + m^2)} \cdot \int_0^1 d\xi\,\delta'(k_1^2\xi + (p_1^2 + m^2)(1-\xi)) \cdot$$

$$\cdot \int_0^1 d\eta\,\delta'(k_2^2\eta + (p_4^2 + m^2)(1-\eta)) = \frac{1}{(2\pi)^6}\iint dq\,dq'\,e^{ix(q'-q)} \cdot$$

$$\cdot \iint dk\,dk'\,\bar{u}(q)\,\gamma_{\lambda_1}(i\gamma(q+k) - m)\,\gamma_{\lambda_1}(i\gamma q - m)\cdot\gamma_\mu\cdot$$

$$\cdot (\mathrm{i}\gamma q' - m)\,\gamma_{\lambda_2}(i\gamma(q'+k') - m)\,\gamma_{\lambda_2}u(q')\,\frac{1}{(q^2+m^2)(q'^2+m^2)}\int_0^1 d\xi\,\delta'(k^2\xi +$$

$$+ ((q+k)^2 + m^2)(1-\xi))\int_0^1 d\eta\,\delta'(k'^2\eta + ((q'+k')^2 + m^2)(1-\eta)) \qquad (98)$$

where

$$u(q) = \frac{1}{(2\pi)^4}\int e^{-iqx}\,\psi^{(0)}(x)\,dx. \qquad (98\text{ a})$$

Introducing the notation

$$K(q) = -\int_0^1 d\xi \int dk\,\gamma_\lambda(i\gamma(q+k) - m)\,\gamma_\lambda\,\delta'(k^2\xi + ((q+k)^2 + m^2)(1-\xi)) \qquad (99)$$

we obtain for the expression (98)

$$\left(\frac{1}{2\pi}\right)^6 \iint dq\,dq'\,e^{ix(q'-q)}\,\bar{u}(q)\,K(q)\,\frac{i\gamma q - m}{q^2+m^2}\,\gamma_\mu\,\frac{i\gamma q' - m}{q'^2+m^2}\,K(q')\,u(q'). \qquad (100)$$

Using the method of DYSON, we obtain instead of the expression (95) the formula

$$\left(\frac{i}{2}\right)^4 \int\cdots\int dx' \ldots dx''''\,\bar{\psi}^{(0)}(x')\,\gamma_\lambda S_F(x'-x'')\,\gamma_\lambda S_F(x''-x)\cdot$$

$$\cdot\gamma_\mu S_F(x-x''')\,\gamma_\varrho S_F(x'''-x'''')\,\gamma_\varrho\,\psi^{(0)}(x''''). \qquad (101)$$

With the aid of the integral representations of FEYNMAN[14]

$$\frac{1}{a_1 \ldots a_n} = (n-1)!\int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-2}} dx_{n-1}\cdot$$

$$\cdot\,[a_n x_{n-1} + a_{n-1}(x_{n-2} - x_{n-1}) + \cdots + a_1(1-x_1)]^{-n} \qquad (102)$$

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[14] R. P. FEYNMAN, Phys. Rev. *76*, 785 (1949). Compare also R. JOST–J. M. LUTTINGER, Helv. Phys. Acta *23*, 208 (1950).

this expression is, in momentum space, conveniently written in the form (100), with $K(q)$ given by

$$K(q) = \frac{1}{i\pi}\int_0^1 d\xi \int_{C_F} dk \, \frac{\gamma_\lambda (i\gamma(q+k) - m)\gamma_\lambda}{k^2\xi + ((q+k)^2 + m^2)(1-\xi)}. \tag{103}$$

Introducing the well-known translation of the origin $k \to k + q(1-\xi)$ and observing that

$$\int dk \, \delta^{(n)}(k^2 + A) f(k^2) = Re \int_{C_F} \frac{dk \, f(k^2)}{(k^2 + A)^{n+1}} \, \frac{(-1)^n n!}{i\pi} \tag{104}$$

we immediately infer, that the real part of (103) is equal to the expression (99). From a comparison of equations (96), (102) and (104) we further conclude, that this equality is not limited to the case of only two lines in the loop but quite general. If A is never zero in the interval $0 \leqq \xi \leqq 1$, the right-hand side of (104) is real, and the two kernels obtained are identical. However, if A does vanish in the interval in question, the integral ($n \geqq 2$)

$$\int_{C_F} \frac{dk}{(k^2 + A)^{n+1}} \tag{105}$$

becomes infinite as[14]

$$\frac{i\pi^2}{n!}\left[(n-2)! \, P\frac{1}{A^{n-1}} + (-1)^n i\pi \, \delta^{(n-2)}(A)\right]. \tag{106}$$

As

$$\int dk \, \delta^{(n)}(k^2 + A) \tag{107}$$

gives only the real part of the right-hand side of (104), we have

$$\int dk \, \delta^{(n)}(k^2 + A) = (-1)^n \pi (n-2)! \, P\frac{1}{A^{n-1}}. \tag{108}$$

The vanishing of A will be related to the occurrence of real processes[15] and gives no trouble in the simple form of diagrams investigated up till now. Both the formalism of DYSON and the integration method developed here are only applicable, when the elements of the S-matrix are effectively equal to one, and it is not astonishing that they will give different results, when this condition is not fulfilled.

---

[15] Compare DYSON[3] p. 1744.

The method of parametrization used to obtain (100) from (101) with the aid of (102) has been found convenient, as it is then especially simple to identify the divergences occurring in the theory.[3] It is, however, not impossible to apply (102) in other ways too. At first it would possibly seem natural to use only one set of auxiliary variables and write for (101) *e.g.*

$$-\frac{1}{\pi^2}\left(\frac{1}{2\pi}\right)^6 \int\int d\,q\,d\,q'\,\bar{u}\,(q)\,K\,(q,\,q')\,u\,(q')\,e^{i\,x\,(q'-q)} \tag{109}$$

with

$$K\,(q,\,q') = 5!\int_0^1 d\,x_1 \int_0^{x_1} d\,x_2 \ldots \int_0^{x_4} d\,x_5 \int\int d\,k\,d\,k'\,\gamma_\lambda\,(i\,\gamma\,(q+k)-m)\cdot$$
$$\cdot\,\gamma_\lambda\,(i\,\gamma\,q-m)\,\gamma_\mu\,(i\,\gamma\,q'-m)\,\gamma_\varrho\,(i\,\gamma\,(q'+k')-m)\,\gamma_\varrho\cdot$$
$$\cdot\,\{[(q'+k')^2+m^2]\,x_5+(q'^2+m^2)\,(x_4-x_5)+(q^2+m^2)\,(x_3-x_4)+$$
$$+\,((q+k)^2+m^2)\,(x_2-x_3)+k^2\,(x_1-x_2)+k'^2\,(1-x_1)\}^{-6}. \tag{110}$$

In this formulation, however, the divergences are not so easily separated as before. In our formalism, on the other hand, only the parametrization given in (98) is natural, and this choice of the auxiliary variables corresponds exactly to the parametrization of equations (100) and (103).

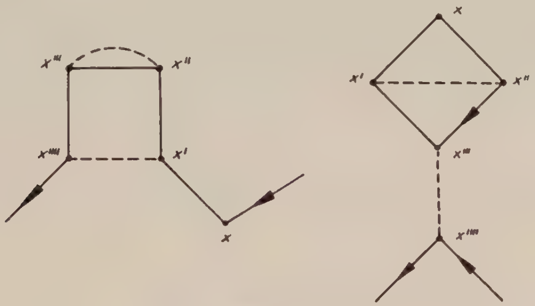

Fig. 12.

A slightly more complicated situation will arise, when two of the loops in the graph have one line in common. As simple examples of such diagrams we can use the two graphs in figure 12, both of which belong to the one-particle part of $j_\mu^{(4)}(x)$. From the calculations given earlier it can be seen without difficulties, that the expectation value of the ${}^{(1)}j_\mu^{(n)}(x)$ operator will give a sum of terms, each containing one $S^{(1)}(x)$ or $D^{(1)}(x)$ function in every loop. Each possible combination will occur, and *e.g.* for the first graph of figure 12 we obtain

$$\begin{aligned}\int\cdots\int dx' \dots dx'''' \,\bar{\psi}^{(0)}(x'''')\,\gamma_\lambda \{&\bar{S}(x''''-x''')\,\gamma_\nu \bar{S}(x'''-x'')\,\gamma_\nu \bar{S}(x''-x')\,\cdot\\ &\cdot D^{(1)}(x''''-x')\,D^{(1)}(x'''-x'')+\bar{S}(x''''-x''')\,\gamma_\nu \bar{S}(x'''-x'')\,\cdot\\ &\cdot \gamma_\nu S^{(1)}(x''-x')\,\bar{D}(x''''-x')\,D^{(1)}(x'''-x'')+\bar{S}(x''''-x''')\,\gamma_\nu\,\cdot\\ &\cdot S^{(1)}(x'''-x'')\,\gamma_\nu \bar{S}(x''-x')\,\bar{D}(x''''-x')\,D^{(1)}(x'''-x'')+\\ &+S^{(1)}(x''''-x''')\,\gamma_\nu \bar{S}(x'''-x'')\,\gamma_\nu \bar{S}(x''-x')\,\bar{D}(x''''-x')\,\cdot\\ &\cdot D^{(1)}(x'''-x'')+\bar{S}(x''''-x''')\,\gamma_\nu S^{(1)}(x'''-x'')\,\gamma_\nu S^{(1)}(x''-x')\,\cdot\\ &\cdot \bar{D}(x''''-x')\,\bar{D}(x'''-x'')+S^{(1)}(x''''-x''')\,\gamma_\nu S^{(1)}(x'''-x'')\,\cdot\\ &\cdot \gamma_\nu \bar{S}(x''-x')\,\bar{D}(x''''-x')\,\bar{D}(x'''-x'')\}\,\gamma_\lambda \bar{S}(x'-x)\,\gamma_\mu \psi^{(0)}(x)\,. \qquad (111)\end{aligned}$$

In these cases we also have contributions from the terms containing the operators ${}^{(2)}\psi^{(n)}(x)$ and ${}^{(2)}A_\mu^{(n)}(x)$. The expressions obtained in that way will contain one functions $S(x)$ or $D(x)$ in the line common to the two loops and another function of that kind in one of the other lines in the loop, not directly connected to the point x. Furthermore, those terms will appear with a minus-sign. For the graph belonging to equation (111) we get simply

$$\begin{aligned}-\int\cdots\int dx' \dots dx'''' \,\bar{\psi}^{(0)}(x'''')\,\gamma_\lambda \bar{S}(x''''-x''')\,\gamma_\nu S(x'''-x'')\,\gamma_\nu\,\cdot&\\ \cdot \bar{S}(x''-x')\,\gamma_\lambda \bar{S}(x'-x)\,\gamma_\mu \psi^{(0)}(x)\,\bar{D}(x''''-x')\,D(x'''-x'')\,. &\qquad (112)\end{aligned}$$

The parametrization of the singular functions in momentum space can now be performed in the following way. For the terms containing a $S^{(1)}(x)$ or $D^{(1)}(x)$ function entirely in the loop directly connected to x, we combine all the expressions with this singular function in the same line with the aid of formula (96). If the momentum variable in the line in common is called p, and the corresponding variables in the loop not connected to x are $p+q'$. $p+q''$, $\dots$ $p+q^n$, we obtain in this way

$$(-1)^n \int dp \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \dots \int_0^{\xi_{n-1}} d\xi_n\, \delta^{(n)}((p+Q)^2+A) \qquad (113)$$

where

$$Q = q^n(\xi_{n-1}-\xi_n)+q^{n-1}(\xi_{n-2}-\xi_{n-1})+\cdots+q'(1-\xi_1) \qquad (113\,\mathrm{a})$$

and

$$A = m^2+q^{n\,2}(\xi_{n-1}-\xi_n)+\cdots+q'^{\,2}(1-\xi_1)-Q^2\,. \qquad (113\,\mathrm{b})$$

The whole expression (113) is multiplied by

$$\begin{aligned}\frac{\delta(p'^{\,2})}{p''^{\,2}(p'''^{\,2}+m^2)\dots(p^{i2}+m^2)}+\frac{\delta(p''^{\,2})}{p'^{\,2}(p'''^{\,2}+m^2)\dots(p^{i2}+m^2)}+\cdots&\\ \cdots+\frac{\delta(p^{i2}+m^2)}{p'^{\,2}p''^{\,2}\dots((p^{i-1})^2+m^2)} &\qquad (114)\end{aligned}$$

where $p', \ldots p^i$ are the momentum variables of the loop connected to the point x. We have here made the assumption, that all the lines in the two loops whith the exception of two are electron lines. If some more of them are photon lines, equations (113 b) and (114) will be slightly changed, but as the modifications necessary are quite trivial, we will not investigate this possibility further. The remaining terms from ${}^{(1)}j_\mu^{(N)}(x)$ can obviously be written

$$(-1)^n \int d p\, \delta(p^2 + m^2) \int_0^1 d\eta_1 \ldots \int_0^{\eta_{n-2}} d\eta_{n-1}\, \delta^{(n-1)}((p + Q')^2 + A') \qquad (115)$$

where

$$Q' = q^n\, \eta_{n-1} + q^{n-1}(\eta_{n-2} - \eta_{n-1}) + \cdots + q'(1 - \eta_1) \qquad (115\text{ a})$$

and

$$A' = q^{n\,2}\, \eta_{n-1} + q^{n-1\,2}(\eta_{n-2} - \eta_{n-1}) + \cdots + q'^{\,2}(1 - \eta_1) + m^2 - Q'^{\,2}. \quad (115\text{ b})$$

The terms from ${}^{(2)}j_\mu^{(N)}(x)$ can analogously be combined in the following way

$$(-1)^n \int_0^1 d\eta_1 \ldots \int_0^{\eta_{n-2}} d\eta_{n-1} \int d p\, \delta(p^2 + m^2)\, \varepsilon(p) \cdot$$

$$\cdot\, \delta^{(n-1)}((p + Q')^2 + A')\, \varepsilon(p + Q') \quad (116)$$

with the same notations A' and Q' as before. The expression (116) is, if no real processes can take place ($A' > 0$), equal to

$$-\frac{(n-1)!}{\pi^2} \int_0^1 d\eta_1 \ldots \int_0^{\eta_{n-2}} d\eta_{n-1}\, P \int \frac{d p}{(p^2 + m^2)((p + Q')^2 + A')^n} \qquad (117)$$

and thus (115) and (116) will combine to

$$(-1)^{n-1} \int_0^1 d\eta_1 \ldots \int_0^{\eta_{n-2}} d\eta_{n-1} \int d p \left\{ \delta(p^2 + m^2)\, \delta^{(n-1)}((p + Q')^2 + A') - \right.$$

$$\left. - \frac{1}{\pi^2} \frac{(-1)^{n-1}(n-1)!}{(p^2 + m^2)((p + Q')^2 + A')^n} \right\}. \qquad (118)$$

Using the well-known integral representations

$$\delta(a) = \frac{1}{2\pi} \int d w\, e^{i w a} \qquad (119)$$

$$P \frac{1}{a} = \frac{1}{2i} \int d w \frac{w}{|w|} e^{i w a} \qquad (120)$$

we now obtain

$$\frac{(-1)^{n-1}}{4\pi^2}\int\limits_0^1 d\eta_1 \dots \int\limits_0^{\eta_{n-2}} d\eta_{n-1} \iint dw_1\, dw_2 \int dp\, e^{iw_1(p^2+m^2)\div iw_2((p+Q')^2+A')}\,.$$

$$\cdot (i w_2)^{n-1}\left(1+\frac{w_1}{|w_1|}\cdot\frac{w_2}{|w_2|}\right) = \frac{(-i)^{n-1}}{2\pi^2}\int\limits_0^1 d\eta_1 \dots \int\limits_0^{\eta_{n-2}} d\eta_{n-1}\,\cdot$$

$$\cdot \int\limits_0^1 d\alpha\,(1-\alpha)^{n-1}\int dw\, w^{n-1}|w| \int dp\, e^{iw[p^2+2pQ'(1-\alpha)+m^2\alpha+(A'+Q'^2)(1-\alpha)]} =$$

$$= -\tfrac{1}{2}(-i)^n \int\limits_0^1 d\alpha\,(1-\alpha)^{n-1}\int\limits_0^1 d\eta_1 \dots \int\limits_0^{\eta_{n-2}} d\eta_{n-1} \int dw\, w^{n-2} e^{iwB} =$$

$$= (-1)^n\cdot\pi \int\limits_0^1 d\alpha\,(1-\alpha)^{n-1}\int\limits_0^1 d\eta_1 \dots \int\limits_0^{\eta_{n-2}} d\eta_{n-1}\,\delta^{(n-2)}(B) \tag{121}$$

where

$$B = m^2 - Q'^2(1-\alpha)^2 + (A'-m^2+Q'^2)(1-\alpha) = m^2 + (q^{n\,2}\eta_{n-1} + \cdots$$
$$\cdots + q'^2(1-\eta_1))(1-\alpha) - Q'^2(1-\alpha)^2\,. \tag{121 a}$$

In the derivation of (121) the transformation

$$\begin{aligned} w_1 &= w\alpha \\ w_2 &= w(1-\alpha) \end{aligned} \tag{121 b}$$

has been used. The new variable-transformation

$$\begin{aligned} \xi_{n-1} - \xi_n &= (1-\alpha)\,\eta_{n-1} \\ \xi_{n-2} - \xi_{n-1} &= (1-\alpha)(\eta_{n-2}-\eta_{n-1}) \\ &\cdots\cdots\cdots\cdots\cdots \\ \xi_2 - \xi_1 &= (1-\alpha)(\eta_2-\eta_1) \\ 1 - \xi_1 &= (1-\alpha)(1-\eta_1) \end{aligned} \tag{122}$$

will change B to A of (113 b) and $Q'(1-\alpha)$ to Q of (113 a). The expression (121) can then be written

$$(-1)^n\pi \int\limits_0^1 d\xi_1 \dots \int\limits_0^{\xi_{n-1}} d\xi_n\,\delta^{(n-2)}(A)\,. \tag{123}$$

The p-integration in (113) can be performed according to well-known rule and we obtain

$$\int d\,p\,\delta^{(n)}((p+Q)^2+A)=\frac{(-1)^{n-2}(n-2)!}{A^{n-1}}\pi. \tag{124}$$

The expression (123) is multiplied by

$$[p'^{\,2}\cdot p''^{\,2}\cdot(p'''^{\,2}+m^2)\dots(p^{i\,2}+m^2)]^{-1} \tag{125}$$

where $p', \dots p^i$ have the same meaning as in (114). Equations (114), (124), (123) and (125) thus combine to give

$$\pi\int_0^1 d\,\xi_1\dots\int_0^{\xi_{n-1}} d\,\xi_n\left\{\left[\frac{\delta(p'^{\,2})}{p''^{\,2}\dots(p^{i\,2}+m^2)}+\dots+\frac{\delta(p^{i\,2}+m^2)}{p'^{\,2}\dots((p^{i-1})^2+m^2)}\right]\cdot\right.$$
$$\left.\cdot\frac{(n-2)!}{A^{n-1}}+\frac{(-1)^n}{p'^{\,2}\dots(p^{i\,2}+m^2)}\,\delta^{(n-2)}(A)\right\}. \tag{126}$$

With a simple modification of (96) (some of the numbers a_i equal) we can introduce a new set of auxiliary variables and perform the remaining momentum variable integrations without any new difficulties of a fundamental nature appearing. The parametrization obtained in this way again corresponds to the most convenient one used by DYSON[3], and the result of the integration is equal to the real part of the formula obtained with the use of the $S_F(x)$ and $D_F(x)$ functions.

At this point it is perhaps not necessary to go further and to write down explicitly the expectation values, when the loops have more than one line in common, or when more than two loops are mixed together. Also in these cases we get one term of the form A^{-n} from the "normal" terms in the loops and contributions containing $\delta^{(n-1)}(A)$ and $\delta^{(n-1)}(A)\cdot\varepsilon(Q)$ from the others. The last expression will appear, when more than two loops are mixed together. We then get terms of the following form

$$\int d\,p\,\delta(p^2+m^2)\,\delta^{(n-1)}((p+Q')^2+A')\,(\varepsilon(p+Q')-\varepsilon(p)) \tag{116'}$$

Comparing (115), (116) and (121), we conclude

$$\int d\,p\,\delta(p^2+m^2)\,\delta^{(n-1)}((p+Q')^2+A')\,(1-\varepsilon(p)\,\varepsilon(p+Q'))=$$
$$=\pi\int_0^1 d\,\alpha\,(1-\alpha)^{n-1}\,\delta^{(n-2)}(A'(1-\alpha)+m^2\,\alpha+Q'^{\,2}\,\alpha\,(1-\alpha)). \tag{121'}$$

This result can be interpreted geometrically. To get a contribution from the left-hand side of (121'), we must have

$$p^2 + m^2 = 0$$

$$(p^2 + Q')^2 + A' = 0$$

$$\frac{p_0}{|p_0|} \cdot \frac{p_0 + Q_0'}{|p_0 + Q_0'|} = -1 .$$

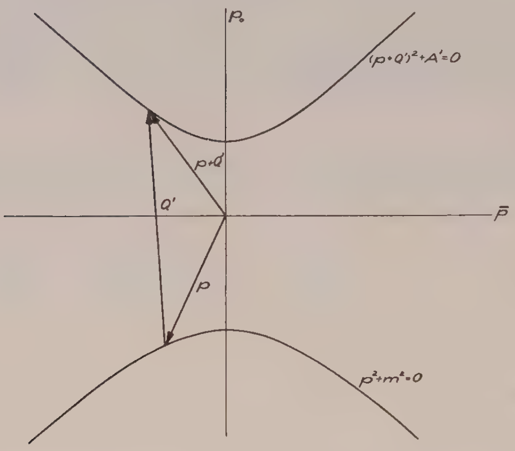

Fig. 13.

These equations cannot be fulfilled at the same time, if we do not have (compare figure 13)

$$|Q_0'| > m + \sqrt{A'}$$

in *all* coordinate systems. This can be written in an invariant way as

$$Q'^{\,2} > -(m + \sqrt{A'})^2$$

which is exactly the condition, that

$$A'(1-\alpha) + m^2 \alpha + Q'^{\,2} \alpha (1-\alpha)$$

will vanish for some value of α in the interval (0, 1). Now, the integral (116′) gets contributions from exactly the same values of p as the integral (121′), and the only difference between the two expressions lies in the signs. We thus obtain

$$\int dp\, \delta(p^2 + m^2)\, \delta^{(n-1)}((p + Q')^2 + A')\, (\varepsilon(p + Q') - \varepsilon(p)) =$$

$$= \pi \int_0^1 d\alpha\, (1-\alpha)^{n-1}\, \varepsilon(Q')\, \delta^{(n-2)}(A'(1-\alpha) + m^2 \alpha + Q'^{\,2} \alpha (1-\alpha)). \qquad (121'')$$

The restrictions on the graphs introduced above are in no way essential for the argument and were only established in order that we should not be burdened with too complicated formulae. It seems to be a general property of the higher approximations of the present form of quantum theory that, although they can be comparatively simply understood from a general point of view, they give nevertheless rise to very cumbersome computations, if we try to make an explicit calculation for a special problem.

It now remains to include the mass-renormalization terms of (55) in the calculation. This can be done comparatively simply by noting that δm in the equation

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi(x) = \frac{i e}{2} \{A_\mu(x), \gamma_\mu \psi(x)\} + \delta m \, \psi(x) \tag{127}$$

plays a role similar to that of $i e \gamma_\mu A_\mu(x)$. Introducing a new vector

$$A'_\mu(x) = A_\mu(x) - \frac{i}{4 e} \gamma_\mu \cdot \delta m \tag{128}$$

we have

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi(x) = \frac{i e}{2} \{A'_\mu(x), \gamma_\mu \psi(x)\} \tag{129}$$

$$\Box A'_\mu(x) = -\frac{i e}{2} [\bar{\psi}(x), \gamma_\mu \psi(x)]. \tag{130}$$

We thus obtain the solution to (127), if we substitute the *total* δm (including all powers of e) in one or more of the places for the operators $i e \gamma_\nu A_\nu^{(0)}(x)$ in the solution of the equation without mass-renormalization term. These substitutions have to be made in all possible ways and the results added together. We will not investigate this point further, as the calculation would be merely a repetition of the argument given by DYSON.

6. Particles with integer spin

In (I) the interaction of particles of spin zero with an external electromagnetic field was studied in some detail. It was, however, also observed, that no serious complications will arise, if the electromagnetic field is quantized. In this case we have the following system of equations

$$(\Box - m^2) \varphi(x) = \frac{i e}{2} \left(\left\{ A_\nu(x), \frac{\partial \varphi(x)}{\partial x_\nu} \right\} + \frac{\partial}{\partial x_\nu} \{A_\nu(x), \varphi(x)\} \right) + \\ + \frac{e^2}{2} \{A_\nu(x) A_\nu(x), \varphi(x)\} \tag{131}$$

$$\Box A_\mu(x) = \frac{i e}{2} \left\{ \left(\varphi^*(x) \frac{\partial}{\partial x_\mu} - \frac{\partial \varphi^*(x)}{\partial x_\mu} \right), \varphi(x) \right\} + e^2 A_\mu(x) \{\varphi^*(x), \varphi(x)\}. \tag{132}$$

Expanding in powers of e as before, we obtain for the first few approximations (compare (I) p. 190)

$$\varphi^{(1)}(x) = -i \int A_\nu^{(0)}(x') \left(\overline{\triangle}(x-x') \frac{\partial}{\partial x'_\nu} + \frac{\partial \overline{\triangle}(x-x')}{\partial x_\nu} \right) \varphi^{(0)}(x')\, d\,x' \tag{133}$$

$$A_\mu^{(1)}(x) = -\frac{i}{2} \int \bar{D}(x-x') \left\{ \left(\varphi^{(0)*}(x') \frac{\partial}{\partial x'_\nu} - \frac{\partial \varphi^{(0)*}(x')}{\partial x'_\nu} \right),\ \varphi^{(0)}(x') \right\} \delta_{\mu\nu}\, d\,x' \tag{134}$$

$$\begin{aligned} \varphi^{(2)}(x) = (-i)^2 \int\!\!\int d\,x'\, d\,x'' \Bigg[A(1,2) \Bigg\{ \Bigg[\overline{\triangle}(x-x') \frac{\partial}{\partial x'_{\nu_1}} + \frac{\partial \overline{\triangle}(x-x')}{\partial x_{\nu_1}} \Bigg] \cdot \\ \cdot \Bigg[\overline{\triangle}(x'-x'') \frac{\partial}{\partial x''_{\nu_2}} + \frac{\partial \overline{\triangle}(x'-x'')}{\partial x'_{\nu_2}} \Bigg] + \overline{\triangle}(x-x')\, \delta(x'-x'')\, \delta_{\nu_1 \nu_2} \Bigg\} \varphi^{(0)}(x'') + \\ + \delta_{\nu_1 \nu_2} \bar{D}(x'-x'') \Bigg[\overline{\triangle}(x-x') \frac{\partial}{\partial x'_{\nu_1}} + \frac{\partial \overline{\triangle}(x-x')}{\partial x_{\nu_1}} \Bigg] \tfrac{1}{4} \cdot \\ \cdot \Bigg\{ \varphi^{(0)}(x'), \Bigg\{ \Bigg(\varphi^{(0)*}(x'') \frac{\partial}{\partial x''_{\nu_2}} - \frac{\partial \varphi^{(0)*}(x'')}{\partial x''_{\nu_2}} \Bigg), \varphi^{(0)}(x'') \Bigg\} \Bigg\} \Bigg] \end{aligned} \tag{135}$$

$$\begin{aligned} A_\mu^{(2)}(x) = (-i)^2 \int\!\!\int d\,x'\, d\,x''\, \tfrac{1}{2} \Bigg[\Bigg\{ \Bigg[\varphi^{(0)*}(x') \frac{\partial}{\partial x'_{\nu_1}} - \frac{\partial \varphi^{(0)*}(x')}{\partial x'_{\nu_1}} \Bigg], \\ \cdot \Bigg[\overline{\triangle}(x'-x'') \frac{\partial}{\partial x''_{\nu_2}} + \frac{\partial \overline{\triangle}(x'-x'')}{\partial x'_{\nu_2}} \Bigg] \varphi^{(0)}(x'') \Bigg\} + \\ + \delta(x'-x'')\, \delta_{\nu_1 \nu_2} \{ \varphi^{(0)*}(x'), \varphi^{(0)}(x'') \} \Bigg] \cdot [\bar{D}(x-x')\, \delta_{\mu\nu_1} A_{\nu_2}^{(0)}(x'') + \\ + \bar{D}(x-x'')\, \delta_{\mu\nu_2} A_{\nu_1}^{(0)}(x')]. \end{aligned} \tag{136}$$

It is clear, that the general structure of the solutions to (131) and (132) is, apart from some differences in signs, very similar to that of the solutions to (78) and (79). We have only to introduce expressions of the form

$$\begin{aligned} \Bigg[\triangle(x-x') \frac{\partial}{\partial x'_{\nu_1}} + \frac{\partial \overline{\triangle}(x-x')}{\partial x_{\nu_1}} \Bigg] \cdots \Bigg[\overline{\triangle}(x^{n-1}-x^n) \frac{\partial}{\partial x^n_{\nu_n}} + \frac{\partial \overline{\triangle}(x^{n-1}-x^n)}{\partial x^{n-1}_{\nu_n}} \Bigg] + \\ + \sum_{i=1}^{n-1} \Bigg[\overline{\triangle}(x-x') \frac{\partial}{\partial x'_{\nu_1}} + \frac{\partial \overline{\triangle}(x-x')}{\partial x_{\nu_1}} \Bigg] \cdots \\ \cdots \Bigg[\overline{\triangle}(x^{i-2}-x^{i-1}) \frac{\partial}{\partial x^{i-1}_{\nu_{i-1}}} + \frac{\partial \overline{\triangle}(x^{i-2}-x^{i-1})}{\partial x^{i-2}_{\nu_{i-1}}} \Bigg] \cdot \\ \cdot \overline{\triangle}(x^{i-1}-x^i)\, \delta_{\nu_i \nu_{i+1}}\, \delta(x^i - x^{i+1}) \cdot \end{aligned}$$

$$\cdot\left[\overline{\triangle}\,(x^{i+1}-x^{i+2})\,\frac{\partial}{\partial\, x^{i+2}_{\nu_{i+2}}}+\frac{\partial\,\overline{\triangle}\,(x^{i+1}-x^{i+2})}{\partial\, x^{i+1}_{\nu_{i+2}}}\right]\cdots$$

$$\cdots\left[\overline{\triangle}\,(x^{n-1}-x^{n})\,\frac{\partial}{\partial\, x^{n}_{\nu_n}}+\frac{\partial\,\overline{\triangle}\,(x^{n-1}-x^{n})}{\partial\, x^{n-1}_{\nu_n}}\right]+\sum_{i<j}\cdots+\cdots \tag{137}$$

(compare (I) equations (17) and (18)) instead of the factors

$$\bar{S}\,(x-x')\,\gamma_{\nu_1}\bar{S}\,(x'-x'')\,\gamma_{\nu_2}\ldots\gamma_{\nu_{n-1}}\bar{S}\,(x^{n-1}-x^{n})\,\gamma_{\nu_n} \tag{138}$$

and

$$\left\{\left[\varphi^{(0)*}\,(x')\,\frac{\partial}{\partial\, x'_{\nu_1}}-\frac{\partial\,\varphi^{(0)*}\,(x')}{\partial\, x'_{\nu_1}}\right],\left[\overline{\triangle}\,(x'-x'')\,\frac{\partial}{\partial\, x''_{\nu_2}}+\frac{\partial\,\overline{\triangle}\,(x'-x'')}{\partial\, x'_{\nu_2}}\right]\cdots\right.$$

$$\left.\cdots\left[\overline{\triangle}\,(x^{n-1}-x^{n})\,\frac{\partial}{\partial\, x^{n}_{\nu_n}}+\frac{\partial\,\overline{\triangle}\,(x^{n-1}-x^{n})}{\partial\, x^{n-1}_{\nu_n}}\right]\varphi^{(0)}\,(x^n)\right\}+$$

$$+\sum_{i=1}^{n-1}\left\{\left[\varphi^{(0)*}\,(x')\,\frac{\partial}{\partial\, x'_{\nu_1}}-\frac{\partial\,\varphi^{(0)*}\,(x')}{\partial\, x'_{\nu_1}}\right],\ldots\overline{\triangle}\,(x^{i-1}-x^{i})\,\cdot\right.$$

$$\left.\cdot\,\delta_{\nu_i\nu_{i+1}}\;\delta\,(x^{i}-x^{i+1})\,\ldots\,\varphi^{(0)}\,(x^n)\right\}+\sum_{i<j}\cdots+\cdots \tag{139}$$

instead of

$$[\bar{\psi}^{(0)}\,(x'),\,\gamma_{\nu_1}\bar{S}\,(x'-x'')\ldots\bar{S}\,(x^{n-1}-x^{n})\,\gamma_{\nu_n}\,\psi^{(0)}\,(x^n)]. \tag{140}$$

(The modifications necessary in (139) for i equal to one are obvious and, for a special case, written down explicitly in (136)). The analysis of the higher approximations is now identical for particles of spin $\frac{1}{2}$ and spin 0 and will not be repeated again.

It is obvious that an argument similar to that given above can be applied to all kinds of mesons in interaction with nucleons or an electromagnetic field. If only the operator equations are covariant, we will always get an invariant result out of the calculation without any elimination of "surface terms". As another simple example we can take the case of pseudoscalar mesons (operator $\varphi\,(x)$, mass m) in vector interaction with nucleons (operator $\psi\,(x)$, mass M). The differential equations are here

$$\left(\gamma\,\frac{\partial}{\partial\, x}+M\right)\psi\,(x)=\frac{i\,g}{2}\left\{\frac{\partial\,\varphi\,(x)}{\partial\, x_\nu},\,\gamma_5\,\gamma_\nu\,\psi\,(x)\right\} \tag{141}$$

$$(\Box-m^2)\,\varphi\,(x)=\frac{i\,g}{2}\,\frac{\partial}{\partial\, x_\nu}\,[\bar{\psi}\,(x),\,\gamma_5\,\gamma_\nu\,\psi\,(x)] \tag{142}$$

$$\gamma_5=\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4\,. \tag{143}$$

The recurrence formulae corresponding to (78) and (79) are

$$\psi^{(n+1)}(x) = -\frac{ig}{2}\int \bar{S}(x-x') \sum_{m=0}^{n} \left\{ \frac{\partial \varphi^{(m)}(x')}{\partial x'_\nu},\ \gamma_5 \gamma_\nu \psi^{(n-m)}(x') \right\} dx' \tag{144}$$

$$\varphi^{(n+1)}(x) = -\frac{ig}{2}\int \frac{\partial \overline{\triangle}(x-x')}{\partial x_\nu} \sum_{m=0}^{n} [\bar{\psi}^{(m)}(x'),\ \gamma_5 \gamma_\nu \psi^{(n-m)}(x')]\, dx' . \tag{145}$$

It is now evident that we obtain the solutions to (144) and (145) from the solutions to (78) and (79) through the following replacements

$$A_\mu^{(0)}(x) \to \frac{\partial \varphi^{(0)}(x)}{\partial x_\mu} \tag{146 a}$$

$$\gamma_\mu \to \gamma_5 \gamma_\mu \tag{146 b}$$

$$\bar{D}(x'-x'')\,\delta_{\nu_1 \nu_2} \to \frac{\partial^2 \overline{\triangle}(x'-x'')}{\partial x'_{\nu_1}\, \partial x''_{\nu_2}} . \tag{146 c}$$

From equation (146 c) we infer, that the expectation values will, in the n^{th} approximation, contain n *more* momentum variables than the corresponding expressions in quantum electrodynamics. The divergences are thus much more serious in this kind of meson theory than in the case of particles of spin one half in interaction with an electromagnetic field.[16]

The other cases of meson-nucleon interaction can be treated in similar ways.

7. Appendix

In this appendix we will investigate a point, which, although it lies a little outside the main discussion, is very conveniently studied in the HEISENBERG representation. We will not carry through the calculations in all approximations, but restrict ourselves to the e^2-terms. Let us consider the one-particle part of the current operator for particles of spin one half. In the notations used in (I) this expression can be written (compare (I) equations (35)–(37))

$$\begin{aligned} < j_\mu^{(2)}(x) >_1 = {} & \frac{i}{2}\int [\bar{\psi}^{(0)}(x),\ \gamma_\mu \bar{S}(x-x')\,(\phi(x') - \delta m^{(2)} \psi^{(0)}(x'))]_1\, dx' + \\ & + \frac{i}{2}\int [(\bar{\phi}(x') - \delta m^{(2)} \bar{\psi}^{(0)}(x'))\, \bar{S}(x'-x)\,\gamma_\mu,\ \psi^{(0)}(x)]_1\, dx' - \\ & - \frac{i}{4}\int\!\!\int [\bar{\psi}^{(0)}(x'),\ K_\mu(x'-x,\ x-x'')\,\psi^{(0)}(x'')]_1\, dx'\, dx'' + \text{vac. pol.} \end{aligned} \tag{A. 1}$$

The operator $\phi(x)$ is logarithmically divergent but, using a regularization with respect to the photon mass[17], we obtain

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[16] P. T. MATHEWS, Phil. Mag. *41*, 185 (1950).

[17] W. PAULI–F. VILLARS, Rev. Mod. Phys. *21*, 434 (1949). Compare also the paper by JOST and LUTTINGER.[14]

$$\phi(x) = \frac{M^2}{(2\pi)^7} \int\int d\,p\,d\,k \int d\,x' \,(i\gamma p + 2m)\, e^{i(p+k)(x-x')}\, \psi^{(0)}(x') \cdot$$
$$\cdot \left\{ \frac{\delta(p^2+m^2)}{k^2(k^2+M^2)} + \frac{\delta(k^2)}{(p^2+m^2)(k^2+M^2)} + \frac{\delta(k^2+M^2)}{(p^2+m^2)\,k^2} \right\}. \quad \text{(A. 2)}$$

This expression can be calculated according to well-known rules, and we obtain

$$\phi(x) = \phi_1(x) + \phi_2(x) \quad \text{(A. 3)}$$

$$\phi_1(x) = \frac{\pi M^2}{(2\pi)^7} \int_0^1 d\,u \int_0^u d\,v \int d\,q \int d\,x'\, e^{iq(x-x')} \frac{u\,(i\gamma q + m)\,\psi^{(0)}(x')}{q^2 u(1-u) + m^2(1-u) + M^2 v} \quad \text{(A. 4)}$$

$$\phi_2(x) = \frac{\pi M^2}{(2\pi)^7} \int_0^1 d\,u \int_0^u d\,v \int d\,q \int d\,x'\, e^{iq(x-x')} \frac{(2-u)\,\psi^{(0)}(x')}{q^2 u(1-u) + m^2(1-u) + M^2 v}. \quad \text{(A. 5)}$$

As $\psi^{(0)}(x)$ fulfils the DIRAC equation for free particles, we can use

$$q^2 + m^2 = 0 \quad \text{(A. 6)}$$

in the *denominators*, and obtain

$$\phi_1(x) = \frac{1}{8\pi^2} \int_0^1 u\,d\,u \log\left(1 + \frac{M^2 u}{m^2(1-u)^2}\right) \left(\gamma \frac{\partial}{\partial x} + m\right) \psi^{(0)}(x) \quad \text{(A. 7)}$$

$$\phi_2(x) = \frac{m}{8\pi^2} \int_0^1 (2-u)\,d\,u \log\left(1 + \frac{M^2 u}{m^2(1-u)^2}\right) \psi^{(0)}(x). \quad \text{(A. 8)}$$

The infinity (dependence of M) in $\phi_2(x)$ can be cancelled, if we define

$$\delta m^{(2)} = \frac{m}{8\pi^2} \int_0^1 d\,u\,(2-u) \log\left(1 + \frac{M^2 u}{m^2(1-u)^2}\right). \quad \text{(A. 9)}$$

It is *not* permitted to put

$$\left(\gamma \frac{\partial}{\partial x'} + m\right) \psi(x') = 0 \quad \text{(A. 10)}$$

in (A. 7) as this expression is multiplied by $\bar{S}(x-x')$ and integrated over x'. As

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \bar{S}(x-x') = -\,\delta(x-x') \quad \text{(A. 11)}$$

the term

$$\int \bar{S}(x-x')\,\phi_1(x')\,dx' \tag{A. 12}$$

is indeterminate and can be given any value of the form

$$-\xi\frac{1}{8\pi^2}\int_0^1 u\,du\,\log\left(1+\frac{M^2 u}{m^2(1-u)^2}\right)\psi^{(0)}(x) \tag{A. 13}$$

with ξ an arbitrary constant.

The term containing $K_\mu(x'-x;\,x-x'')$ in (A. 1) also contains a divergency. This can be separated in the usual way[18], and we get, after regularization

$$-\frac{1}{4\pi^2}\gamma_\mu\,\delta(x'-x)\,\delta(x-x'')\int_0^1 u\,du\,\log\left(1+\frac{M^2 u}{m^2(1-u)^2}\right). \tag{A. 14}$$

Collecting the terms from (A. 13), its Hermitian conjugate and (A. 14), we obtain

$$\frac{i}{16\pi^2}\int_0^1 du\cdot u\cdot\log\left(1+\frac{M^2 u}{m^2(1-u)^2}\right)(1-2\xi)\,[\bar{\psi}^{(0)}(x),\,\gamma_\mu\psi^{(0)}(x)]_1. \tag{A. 15}$$

The second approximation of the current operator thus contains an indeterminate multiple of the zeroth-approximation current. From a mathematical point of view, this corresponds to a solution of the homogeneous equation added to the expression given *e.g.* in equation (49). Such an extra term ought to have been avoided according to our boundary conditions (5) and the use of the retarded functions. This is, however, not the case and is probably connected with the fact that the operators $\psi^{(0)}(x)$ do not vanish for $x_0 \to -\infty$. Hence, the solutions (49) and (50) do not strictly fulfil (51). According to DYSON[3], this indeterminacy is avoided by observing that the normalization of $\psi^{(0)}(x)$ must not be altered by the electromagnetic interaction. The normalization condition for the electron field is, however, just the time-component of the current operator, and hence (A. 15) must vanish. This means

$$\xi = \tfrac{1}{2}$$

and agrees in the e^2-approximation with the result of DYSON.[19]

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[18] J. SCHWINGER, Phys. Rev. *76*, 790 (1949).

[19] Cf. also R KARPLUS—N. M. KROLL, Phys. Rev. *77*, 542 (1950).

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